EXERCISE BEHAVIOR – GALAI METHOD

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CONTRACT DESCRIPTION

The employee stock option is issued at time t = 0. The expiration date is at time t = Time (years) and the dividend yield is not necessary zero. The option can be exercised at any time on or after t = TimeExer1 (years).

MODEL DESCRIPTION

The rational investor is expected to exercise the option at the optimal stopping time. For call options, when the dividend yield is zero, the optimal stopping time is the expiration date (Hull, 1993a page 158). However, the employee is not expected to behave as a rational investor and is expected to exercise earlier (Rubinstein, 1994). In order to take into account the employee exercise behavior, we consider the possibility of exercise at time t = TimeExer1 (years), t = TimeExer2 (years), … , t = TimeExerN (years), and t = Time (years). For each time when **early** exercise is considered we define underlying intervals (through their upper and lower limits) and attach a probability for early exercise to each underlying interval. For t = Time (years), because it is the expiration date, there is no early exercise. If there is a single interval specified for a given date, then both limits are included in the interval. Otherwise, the lower limit is not included in the interval, while the upper limit is included in the interval. At a given moment, if the underlying falls within the limits of one of the intervals specified above, then the option is exercised with the given probability. If the underlying is outside of the intervals specified above, then there is no early exercise. At maturity all in-the-money contracts are exercised. At maturity, all at-the-money and out-of-money contracts are counted as contracts not exercised. When early exercise is considered, if one of the intervals specified above includes values below the exercise price, and if the underlying is below the exercise price, the contract is "killed" with the given probability.

The option is American, the yield is not always zero, and therefore early exercise may be optimal. Because we check for early exercise on a limited number of dates, the computed value of the option may be significantly below its theoretical value. This error is due to the "irrational" investor who is not checking for early exercise at each moment.

The underlying is supposed to follow a geometric Brownian motion with drift (Black and Scholes,1973; Merton, 1973). Following Black and Scholes (1973) and Merton (1973; 1990), the corresponding stochastic differential equation is integrated using the Ito (1944; 1946; 1951) interpretation. Two approaches are used to model the process: the risk-neutral approach (Hull, 1993a page 221-222) and the Galai (1978) approach.

The risk-neutral approach. In order to estimate the option price, the drift is set equal to the risk-free interest rate (Hull, 1993a page 221-222). However, this risk-neutral approach does not imply that the return rate of the stock equals the risk-free interest rate (Hull, 1993b page 62; Wilmott et al. 1997). Therefore, in order to check if the employee uses early exercise, we have to model the underlying as a geometric Brownian motion with drift equal to its own rate of return. The model adopted simulates at the same time two paths for the underlying: one path using the risk-free interest rate that is necessary to estimate the discounted payoff; and another path using the rate of return of the stock that is necessary to decide when the option is exercised. Both paths are simulated using the same volatility and the same string of random numbers, i.e. the paths are “parallel”.

Tha Galai approach. The underlying has the drift equal to its own rate of return. The discounted value of the payoff is computed using the discount rate suggested by Galai (1978) instead of the risk-free interest rate. Using the Galai approach we have to simulate a single path for the underlying; this path is used to decide whether or not early exercise is warranted and to compute the option value. The Galai approach may be more intuitive for the user. Although both approaches are expected to provide the same option value, the other statistics are different.

There are two alternatives used to generate random numbers: a high-quality reproducible method, and a “quick-and-dirty” non-reproducible method. In the first method the random numbers are generated coupling the “minimal” uniform random number generator of Park and Miller (1988) with the Box-Muller method for Gaussian distribution. The second method relies upon a “seed” for starting the random number generation and is machine dependent. More details regarding the Monte Carlo simulations are provided by Montgomery and Pizzi (1998).

MODEL ACCURACY

The accuracy of the model can be checked “forcing” the employee to exercise on a given date t = t1. If: a) at t = t1 the lower limit for the first underlying interval is zero; b) at t = t1 the upper limit for the first underlying interval is infinity (e.g, 10^{+32}); c) at t = t1the probability for early exercise is 1; and d) at t < t1 the lower limit of the first interval is infinity (e.g, 10^{+32}) and the probability for early exercise is zero, then the employee is “forced” to exercise after t1 (years). In this case the option value is equal to the usual Black-Scholes European option exercised after t1 (years).

The speed of convergence of the Monte Carlo results to the correct values depends on volatility and time: higher volatility and/or time requires a higher number of Monte Carlo simulations in order to achieve the same accuracy.

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