Volatility and Beta

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Motivation

From time to time we will receive a question from one of our valuation clients regarding our estimate of the expected volatility of a stock and the relationship of the estimate to the "beta," which is reported by a number of sources. The purpose of this Technical Note is to inform our Clients and other interested parties as to the precise nature of expected volatility and beta, and to clarify the relationship between the two. Although the beta of a stock is related to the historical volatility of the share price, it is incorrect to use the two terms interchangeably.

Volatility

We will consider this comparison from the perspective of historical volatility, which is computed from share price data over a period of time in the past. Although it is possible, and sometimes advisable, to consider the forward looking implied volatility, computed from the prices of traded options, we choose historical volatility because the "beta" of a stock is also derived from historical data.

Consider an asset **X** with a series of market prices $\mathbf{x_1, x_2, \ldots, x_n}$ recorded at times $\mathbf{t_1, t_2, \ldots, t_n}$. The intervals of time are defined to all be of equal length and each succeeding time is fixed at the same time in each trading session (open or close). In most cases, closing prices are used, although nothing compels us to use the closing price other than convenience. From these price data, a series of returns denoted by $\mathbf{r_i}$ may be computed. If we assume that the return is continuously compounded over each time interval, then:

$$\mathbf{r_i = \ln(x_i/x_{i-1})}$$

Let us say that we now have a series of **N** successive returns over a specific time interval (daily, weekly, monthly, etc.). The mean return $\boldsymbol{\mu}$ is simply the arithmetic mean of the **N** values of the periodic return. When arranged in a progression from smallest to largest over a series of predefined levels, $\mathbf{L_i}$, the returns form a probability distribution over the ascending series of levels. The mean of the distribution is the mean return $\boldsymbol{\mu}$. The variance of the distribution of returns, known as $\boldsymbol{\sigma^2}$, is the sum of the squared deviations from the mean divided by **N-1**:

$$\boldsymbol{\sigma^2 = (1/(N - 1))\Sigma(x_i - \mu)^2}$$

The positive square root of the variance is known as the ***standard deviation*** of the distribution of returns. The ***volatility***, expressed as *percent per annum*, is the standard deviation of the distribution of returns multiplied by the square root of the number of time intervals per year. For example, if returns are calculated on a daily basis, one would multiply the standard deviation by the square root of 252, the number of trading days per year.

Portfolios and Indices

We can generalize this treatment to a portfolio of assets. For each asset in the portfolio, $\mathbf{X_i}$, there is a specific weight, $\boldsymbol{\omega_i}$. The sum of all the weights must equal 1.00. Each weighting factor $\boldsymbol{\omega_i}$ represents the number of units of asset $\mathbf{X_i}$ per unit of the entire portfolio. The value of one unit of the portfolio is the weighted average of the values of the individual assets:

$$\mathbf{P = \Sigma\omega_i x_i}$$

The value of the portfolio may be interpreted as the value of an index comprising a number of assets, each with its own weighting factor. These weighting factors may be fixed, as in the case of an equal allocation portfolio or un-weighted index. Weighting factors are variable in the case of a market capitalization weighted portfolio or index. The S&P 500 is a market cap weighted index, in which the weighting factor of each individual stock is equal to the market cap of the given stock divided by the sum of the market caps of all 500 stocks in the index. The Dow Jones Industrial Average is a price-weighted index. This is the sum of the prices of all 30 stocks divided by the DJIA Divisor, which is a number adjusted for dividend payouts and stock splits to keep the index continuous over time. The Nikkei 225 is another well-known price-weighted index.

For a price-weighted index with a constant divisor, we may calculate the successive returns on the index over a number of time intervals in the like manner as for an individual stock. From this series of returns, we can then calculate the mean, variance, and historical volatility of the index for a specific period of time with a specific data frequency.

The mean return on the index is simply the weighted means of the returns of all the constituent stocks or other assets as the case may be:

$$\mathbf{R = \Sigma\omega_i r_i}$$

The variance and its positive square root (the volatility) of the index depend not only upon the variances of the individual constituent assets and their weighting factors, but also upon the pairwise correlation coefficients between each pair of assets. The two-component embodiment of this relationship is a familiar to students of elementary portfolio theory. For a two-asset portfolio:

$$\boldsymbol{\sigma_P = \sigma_{12} = [\,\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\rho_{12}\sigma_1\sigma_2\,]^{1/2}}$$

This result may be generalized to a case of **N** weighting factors and **N** individual volatilities, where $\boldsymbol{\rho_{ij}}$ is the correlation coefficient of the $\mathbf{i^{th}}$ and $\mathbf{j^{th}}$ random variables, or returns, where:

$$\boldsymbol{\rho_{ij} = \mathrm{Cov}(r_i, r_j)/\sigma_i\sigma_j}$$

Since $\boldsymbol{\rho_{ij} = \rho_{ji}}$ and $\boldsymbol{\rho_{ii} = 1.00}$, each set of **N** assets is characterized by **N** average returns, **N** volatilities, and **N(N+1)/2** unique covariances.

$$\sigma_p^2 = \Sigma\omega_i^2\sigma_i^2 + 2\Sigma\Sigma\omega_i\omega_j\rho_{ij}\sigma_i\sigma_j$$

Working backward from the return and volatility of the index, and given a set of weighting factors, there is not necessarily a unique solution for the individual returns and volatilities of the constituent assets without knowledge of the full set of covariances or correlation coefficients. It is also evident that the volatility of the index will increase if either the volatility of the individual components increases or if the correlation between components increases.

If one considers the implied volatility of an index derived from the prices of traded index options, we note that the index implied volatility contains an implied correlation. In circumstances where the individual component volatilities are unchanging, Changes in correlation will manifest themselves as variation in the volatility of the index.

In the days before widespread use of inexpensive computing power, the calculation of several hundred or several thousand correlation coefficients could be tedious or even prohibitively time-consuming. In such an environment, the approximations embodied in the Capital Asset Pricing Model (CAPM) proved most useful.

Indices and the CAPM

The CAPM models the relationship between the excess return of an individual asset or portfolio as dependent on a simple factor, known as beta, $\boldsymbol{\beta}$. In its simplest form, the CAPM may be stated as:

$$R_i = R_f + \beta_i(R_m - R_f)$$

Note that $\mathbf{R_i}$ is the return of the individual asset, $\mathbf{R_f}$ is the risk-free rate, and $\mathbf{R_m}$ is the "market return" or the overall return of a given index. Using the consequences of elementary portfolio theory, we note that

$$\beta_i = Cov(r_i, r_m)/\sigma_m^2 = \rho_{im}\sigma_i/\sigma_m$$

One may determine the value of $\boldsymbol{\beta_i}$ from observations of the index and the individual asset, without decomposing the index into its component parts. The $\boldsymbol{\beta}$ of an individual asset is representative of the correlation of the individual asset with the entire index, without regard to the correlation of the chosen asset to the individual asset components, or of those components with each other. It is assumed that the "market portfolio" is well-diversified, that is that only systematic risk remains after all individual risks of the component assets have been diversified away. For an index such as the S&P 500, this is a good assumption. For individual sector indices, or for peer groups comprising similar companies, this assumption does not apply.

Conclusions

In this brief note, we have made two points:

1. The behavior of an index does not allow us to gain accurate insight into the behavior of the individual assets which make up the index.
2. Asset correlations, and thus the volatility of an index, may be subject to change even when the volatilities of the individual components remain unchanged.

If one wishes to simply compare the return and risk of an individual firm with respect to the entire market, the beta as defined in the Capital Asset Pricing Model is suitable. For all other purposes, it is necessary to examine the volatilities of all individual assets under consideration. ***Historical beta is of no use in the estimation of either historical or expected volatility.*** The beta metric has its uses in the management of well-diversified portfolios, but is inappropriate for forecasting the performance of individual equities over periods of time in which correlations may change.

June 2013