WARRANT VALUATION METHODS

Sorin R. Straja, Ph.D., FRM
Montgomery Investment Technology, Inc.
200 Federal Street
Camden, NJ 08103
Phone: (610) 688-8111
sorin.straja@fintools.com
www.fintools.com

We consider the case of a European style warrant that entitles the contract holder to buy at the strike price one share of the underlying stock.

BLACK-SCHOLES-MERTON MODEL

The value of the warrant (W) is equal to the value of a call option with the same strike and time-until-expiration:

$$W = BSM(S, X, T, \sigma_S, r, y) \qquad (1)$$

The call option value is estimated using the Black-Scholes-Merton formula (BSM) where S is the stock value, X is the strike, T is the time-until-expiration, σ_S is the stock volatility, r is the risk-free interest rate and y is the dividend yield.

DILUTED BLACK-SCHOLES-MERTON MODEL

The value of the warrant is equal to the diluted value of a call option with the same strike and time-until-expiration:

$$W = \frac{N}{N+n} \cdot BSM(S, X, T, \sigma_S, r, y) \qquad (2)$$

The dilution factor is the number of outstanding stock shares (N) divided by sum of the numbers of outstanding stock shares and issued warrants (n).

Crouhy and Galai (1991a) note that in practice warrant prices are often calculated by multiplying the outcome from the Black-Scholes-Merton formula by the dilution factor. Galai (1989) and Crouhy and Galai (1991a) argue that this procedure, which is based upon a misinterpretation of the Galai and Schneller (1978) model. According to Crouhy and Galai (1991a) this model is inherently wrong:

"This approach is wrong and, in general, leads to an underestimation of the warrant's price. An alternative approach is to determine the value of the warrants, and the firm's other liabilities, simultaneously with the value and the volatility of the firm's assets."

GALAI-SCHNELLER MODEL WITH DIVIDEND YIELD

Denoting by V the company's value, at maturity the payoff for a warrant is (Galai and Schneller 1978; Cox and Rubinstein 1985 pages 394-395; Hull 1993 pages 228-229):

$$\max\left(\frac{V + n \cdot X}{N + n} - X \ , 0\right) \quad = \quad \max\left(\frac{V - N \cdot X}{N + n} \ , 0\right) \quad =$$

$$= \quad \frac{N}{N + n} \cdot \max\left(\frac{V}{N} - X \ , 0\right) \qquad (3)$$

Therefore, the warrant value is equal to the value of a call option (adjusted for dilution) on the company (not stock) value (Galai and Schneller 1978; Crouhy and Galai 1991a; Crouhy and Galai 1991b):

$$W \quad = \quad \frac{N}{N + n} \cdot BSM \quad \left(\frac{V}{N}, \ X \ , T \ , \sigma_C \ , \ r \ , \ 0\right) \qquad (4)$$

In the above equation σ_C is the volatility of the company (not stock) value and V is (Li and Wong 2004; Li and Wong 2005):

$$V \quad = \quad N \cdot S \cdot e^{-y \cdot T} + n \cdot W \qquad (5)$$

Using the formula for the company value it results:

$$W \quad = \quad \frac{N}{N + n} \cdot BSM\ (S \cdot e^{-y \cdot T} + \frac{n}{N} \cdot W, X, T, \sigma_C, r, 0) \qquad (6)$$

Using the Black-Scholes-Merton formula we get:

$$W \quad = \quad \frac{N \cdot S \cdot e^{-y \cdot T} + n \cdot W}{N + n} \cdot \Phi(d_1) - \frac{N \cdot X \cdot e^{-r \cdot T}}{N + n} \cdot \Phi(d_2) \qquad (7)$$

where:

$$d_1 = \frac{\ln\left(\frac{N \cdot S \cdot e^{-y \cdot T} + n \cdot W}{N \cdot X}\right) + (r + \frac{\sigma_C^2}{2}) \cdot T}{\sigma_C \cdot \sqrt{T}} \quad (8)$$

$$d_2 = d_1 - \sigma_C \cdot \sqrt{T} \quad (9)$$

In equation (7) Φ is the cumulative normal distribution function. It should be noted that equation (7) is an implicit equation for the warrant value which appears on both the right and left hand side of the equal sign. The solution of this equation requires iterations.

UKHOV-LI-WONG MODEL WITH DIVIDEND YIELD

Following Ukhov (2003; 2004) and Li and Wong (2004; 2005) the warrant pricing equation (7) can be re-written using the company value instead of the stock value:

$$V = \frac{n \cdot V}{N + n} \cdot \Phi(d_1) - \frac{n \cdot N \cdot X \cdot e^{-r \cdot T}}{N + n} \cdot \Phi(d_2) + N \cdot S \cdot e^{-y \cdot T} \quad (10)$$

where:

$$d_1 = \frac{\ln\left(\frac{V}{N \cdot X}\right) + (r + \frac{\sigma_C^2}{2}) \cdot T}{\sigma_C \cdot \sqrt{T}} \quad (11)$$

$$d_2 = d_1 - \sigma_C \cdot \sqrt{T} \quad (12)$$

Following Ukhov (2003; 2004) from equation (5) we get the following relationship (Ukhov 2003; Ukhov 2004):

$$1 = N \cdot \frac{dS}{dV} \cdot \exp(-y \cdot T) + n \cdot \frac{dW}{dV} \quad (13)$$

From equation (4) we get:

$$\frac{dW}{dV} = \frac{1}{N + n} \cdot \Phi(d_1) \qquad (14)$$

Therefore from equations (13)-(14) it results:

$$\frac{dS}{dV} = \frac{N + n - n \cdot \Phi(d_1)}{N\,(N + n)} \cdot e^{y \cdot T} \qquad (15)$$

Stock volatility is related to firm volatility (Crouhy and Galai 1994; Schulz and Trautmann 1994; Ukhov 2003; Ukhov 2004) as follows (see Appendix):

$$\sigma_S = \sigma_C \cdot \frac{V}{S} \cdot \frac{dS}{dV} \qquad (16)$$

Therefore from equations (15)-(16) it results:

$$V = \frac{\sigma_S}{\sigma_C} \cdot S \cdot e^{-y \cdot T} \cdot \frac{N\,(N + n)}{N + n - n \cdot \Phi(d_1)} \qquad (17)$$

Equations (10) and (17) represent a system of two non-linear equations with two unknowns, V and σ_C. The solution of this system is used in equation (4) to obtain the warrant value.

UKHOV-DAVES-EHRHARDT MODEL WITH DIVIDEND YIELD

Following Ukhov (2003; 2004) and Dave and Ehrhardt (2005 footnote 20) the warrant pricing equation (7) can be re-written as follows:

$$V = \frac{n \cdot V}{N + n} \cdot \Phi(d_1{}^q) - \frac{n \cdot N \cdot X \cdot e^{-r \cdot T}}{N + n} \cdot \Phi(d_2{}^q) + N \cdot S \qquad (18)$$

where :

$$V = N \cdot S + n \cdot W \qquad (19)$$

$$N \cdot S \cdot e^{-y \cdot T} + n \cdot W \quad = \quad (N \cdot S + n \cdot W) \cdot e^{-q \cdot T} \qquad (20)$$

$$d_1^{\,q} \quad = \quad \frac{\ln\left(\dfrac{V}{N \cdot X}\right) + (r - q + \dfrac{\sigma_C^{\,2}}{2}) \cdot T}{\sigma_C \cdot \sqrt{T}} \qquad (21)$$

$$d_2^{\,q} \quad = \quad d_1^{\,q} - \sigma_C \cdot \sqrt{T} \qquad (22)$$

From equation (19) we get the following relationship:

$$1 \quad = \quad N \cdot \frac{\partial S}{\partial V} + n \cdot \frac{\partial W}{\partial V} \qquad (23)$$

The warrant value is equal to the value of a call option (adjusted for dilution) on the company (not stock) value (Galai and Schneller 1978; Crouhy and Galai 1991a; Crouhy and Galai 1991b):

$$W \quad = \quad \frac{N}{N+n} \cdot BSM\left(\frac{V}{N}, X, T, \sigma_C, r, q\right) \qquad (24)$$

It should be noted that from equations (24) and (19)-(22) we get:

$$\frac{\partial W}{\partial V} \quad = \quad \frac{N \cdot e^{-q \cdot T}}{N+n} \cdot \Phi(d_1^{\,q}) \cdot \frac{1}{N} - \frac{N \cdot T \cdot e^{-q \cdot T}}{N+n} \cdot \Phi(d_1^{\,q}) \cdot \frac{V}{N} \cdot \frac{\partial q}{\partial V} =$$

$$= \quad \frac{e^{-q \cdot T}}{N+n} \cdot \Phi(d_1^{\,q}) \cdot \left(1 - T \cdot V \cdot \frac{\partial q}{\partial V}\right) =$$

$$= \quad \frac{e^{-q\cdot T}}{N+n}\cdot\Phi(d_1{}^q)\cdot\left[1+\frac{N\cdot S\cdot\left(1-e^{-y\cdot T}\right)}{V-N\cdot S\cdot\left(1-e^{-y\cdot T}\right)}\right]=$$

$$= \quad \frac{e^{-q\cdot T}}{N+n}\cdot\Phi(d_1{}^q)\cdot\left[\frac{V}{V-N\cdot S\cdot\left(1-e^{-y\cdot T}\right)}\right]=$$

$$= \quad \frac{V-N\cdot S\cdot\left(1-e^{-y\cdot T}\right)}{V}\cdot\frac{1}{N+n}\cdot\Phi(d_1{}^q)\cdot\left[\frac{V}{V-N\cdot S\cdot\left(1-e^{-y\cdot T}\right)}\right]$$

In conclusion we get:

$$\frac{\partial W}{\partial V} \quad = \quad \frac{\Phi(d_1{}^q)}{N+n} \qquad (25)$$

Therefore from equations (23) and (25) it results:

$$\frac{\partial S}{\partial V} \quad = \quad \frac{N+n-n\cdot\Phi(d_1{}^q)}{N\ (N+n)} \qquad (26)$$

From equations (16) and (26) it results:

$$V \quad = \quad \frac{\sigma_S}{\sigma_C}\cdot S\cdot\frac{N\ (N+n)}{N+n-n\cdot\Phi(d_1{}^q)} \qquad (27)$$

Equations (18) and (27) represent a system of two non-linear equations with two unknowns, V and σ_C. The solution of this system is used in equation (24) to obtain the warrant value.

LAUTERBACH-SCHULTZ MODEL WITH DIVIDEND DISTRIBUTIONS

Lauterbach and Schultz (1991) use a version of the Galai and Schneller (1978) model adapted for known dividend distributions. If in equation (7) instead of $S \cdot e^{-yT}$ we use [S-PV(D_i)] where PV(D_i) is the present value of the future dividend distributions, then we obtain the Lauterbach and Schultz (1991) model.

Under some market conditions it may be more realistic to make assumptions regarding the amount of the dividends paid at different dates rather than to assume the value of the dividend yield. In this case the stock price S (with volatility σ_S) can be seen as the sum of two components:

1. One risk-less component corresponding to the known dividends during the life of the contract; and

2. Another continuous risky component S^* (with volatility σ_S^*) with no dividends.

At any given time the risk-less component is the present value of the future dividends. The discounting rate is the risk-free interest rate.

At the beginning of the simulation (t_0), assuming that at time t_i of the dividend payment is D_i, the value of S^* will be:

$$S^*(t_0) = S(t_0) - \sum_{i=1}^{n} D_i \cdot e^{[-(t_i - t_0) \cdot r]} \qquad (28)$$

Hull (1993 page 347) assumes that σ_S^* is constant (not σ_S) and mentions that in general $\sigma_S^* > \sigma_S$. However, Hull (1993) does not indicate how to compute σ_S^*.

Chriss (1997 page 157 equation 4.9.1) suggests the following formula for the case with one dividend distribution:

$$\sigma_S^* = \frac{S(t_0) \cdot \sigma_S}{S(t_0) - D \cdot e^{[-(t_D - t_0) \cdot r]}} \qquad (29)$$

In the above formula D is the dividend distribution at time t_D and t_0 is the present time. This formula is mentioned also by Hull (2003 page 253).

Beneder and Vorst (2001) expand and improve the Chriss (1997) approximation using a weighted average of an adjusted and an unadjusted variance where the weighting depends on the time t_i of the dividend payment D_i:

$$\sigma_S^{*2} \cdot (T - t_0) = \sigma_S^2 \cdot (T - t_n) +$$

$$+ \sum_{j=1}^{n} \left(\frac{S(t_0) \cdot \sigma_S}{S(t_0) - \sum_{i=j}^{n} D_i \cdot e^{[-(t_i - t_0) \cdot r]}} \right)^2 \cdot (t_j - t_{j-1}) \quad (30)$$

In the above formula n is the number of known dividend distributions during the life of the contract and T is the expiration time. Amaro de Matos et al. (2006) report that the Beneder and Vorst (2001) approximation performs significantly better than assuming $\sigma_S^* = \sigma_S$.

In order to use the Lauterbach and Schultz (1991) model, it should be noted that once σ_S^* is known, it is necessary to estimate σ_C, i.e. the volatility of the company (not stock) value.

UKHOV-CHEN-LI MODEL WITH DIVIDEND DISTRIBUTIONS

For the case of known dividend distributions equation (5) becomes (Chen and Li 2008):

$$V = N \cdot \{S(t_0) - \sum_{i=j}^{n} D_i \cdot e^{[-(t_i - t_0) \cdot r]}\} + n \cdot W \quad (31)$$

Following Ukhov (2003; 2004) and Chen and Li (2008) the warrant pricing equation (7) can be re-written as follows:

$$V = \frac{n \cdot V}{N + n} \cdot \Phi(d_1) - \frac{n \cdot N \cdot X \cdot e^{-r \cdot T}}{N + n} \cdot \Phi(d_2) +$$

$$+ N \cdot \{S(t_0) - \sum_{i=j}^{n} D_i \cdot e^{[-(t_i - t_0) \cdot r]}\} \quad (32)$$

where:

$$d_1 = \frac{\ln\left(\frac{V}{N \cdot X}\right) + \left(r + \frac{\sigma_C^2}{2}\right) \cdot T}{\sigma_C \cdot \sqrt{T}} \quad (33)$$

$$d_2 = d_1 - \sigma_C \cdot \sqrt{T} \quad (34)$$

$$W = \frac{N \cdot \{S(t_0) - \sum_{i=j}^{n} D_i \cdot e^{[-(t_i - t_0) \cdot r]}\} + n \cdot W}{N + n} \cdot \Phi(d_1) -$$

$$- \frac{N \cdot X \cdot e^{-r \cdot T}}{N + n} \cdot \Phi(d_2) \qquad (35)$$

Following Ukhov (2003; 2004), from equation (31) we get the following relationship:

$$1 = N \cdot \frac{dS}{dV} + n \cdot \frac{dW}{dV} \qquad (36)$$

From equation (35) we get:

$$\frac{dW}{dV} = \frac{1}{N + n} \cdot \Phi(d_1) \qquad (37)$$

Therefore from equations (36)-(37) it results:

$$\frac{dS}{dV} = \frac{N + n - n \cdot \Phi(d_1)}{N\ (N + n)} \qquad (38)$$

Therefore from equations (16) and (38) it results (Chen and Li 2008):

$$V = \frac{\sigma_S}{\sigma_C} \cdot S \cdot \frac{N\ (N + n)}{N + n - n \cdot \Phi(d_1)} \qquad (39)$$

Equations (32) and (39) represent a system of two non-linear equations with two unknowns, V and σ_C. The solution of this system is used in equation (35) to obtain the warrant value.

It should be noted that when Chriss approximation (equation 29) may be used then it results:

$$V = \frac{\sigma_S^*}{\sigma_C} \cdot \{S(t_0) - D \cdot e^{[-(t_D - t_0) \cdot r]}\} \cdot \frac{N\,(N+n)}{N + n - n \cdot \Phi(d_1)} \qquad (40)$$

Equations (32) and (40) represent a system of two non-linear equations with two unknowns, V and σ_C. This system is similar to equations (10) and (17) where:

1. Instead of $S \cdot e^{-yT}$ we use $[S-PV(D_i)]$ where $PV(D_i)$ is the present value of the future dividend distributions; and

2. Instead of σ_S we use σ_S^*.

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APPENDIX

Let us assume that the stock (S) follows a geometric Brownian motion process:

$$dS = (\mu - y) \cdot S \cdot dt + \sigma_S \cdot S \cdot dw \qquad (A1)$$

According to Ito's Lemma the function V follows the process described below:

$$dV = \left[(\mu - y) \cdot S \cdot \frac{\partial V}{\partial S} + \frac{1}{2} \cdot \sigma_S^2 \cdot S^2 \cdot \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] \cdot dt + $$
$$+ \sigma_S \cdot S \cdot \frac{\partial V}{\partial S} \cdot dw \qquad (A2)$$

This process may be described as follows:

$$dV = \alpha \cdot dt + \sigma_C \cdot V \cdot dw \qquad (A3)$$

Therefore, identifying the coefficients of dw from equations (A2)-(A3), we have the following condition:

$$\sigma_S \cdot S \cdot \frac{\partial V}{\partial S} = \sigma_C \cdot V \qquad (A4)$$

The function V has not been specified. Some examples are listed below:

$$V = n \cdot W + N \cdot S \qquad (A5)$$

$$V = n \cdot W + N \cdot S \cdot e^{-y \cdot T} \qquad (A6)$$

$$V = N \cdot S \qquad (A7)$$