ABSTRACT

The geometric Brownian motion model is widely used to explain the stock price time series. The following sections summarize its main features. The stochastic model may be viewed as an extension of the usual deterministic model for which the rate of return is viewed as a constant value subjected to perturbations. We present both the Itô and Stratonovich interpretations of the resulting stochastic differential equation. The parameters estimation and model predictions could be done using either interpretation; however, the same interpretation must be used for both steps (i.e., parameters estimation and model predictions).

INTRODUCTION

Bachelier (1900) seems to be the first to have provided an analytical valuation for stock options. His work is rather remarkable because by addressing the problem of option pricing, Bachelier (1900) derived most of the theory of diffusion processes. The mathematical theory of Brownian motion has been formulated by Bachelier (1900) five years before Einstein’s classic paper (Einstein 1905). Bachelier (1900) has formulated avant la lettre the Chapman-Kolmogorov equation (von Smoluchowski 1906; Chapman 1916; Chapman 1917; Kolmogorov 1931), called today the Chapman-Kolmogorov-Smoluchowski-Bachelier equation (Brown et al. 1995), and the Fokker-Plank or Kolmogorov equation (von Smoluchowski 1906; Fokker 1914; Fokker 1917; Plank 1917; Kolmogorov 1931). Moreover, the first-passage distribution function for the drift-free case was provided by Bachelier (1900) before Schrödinger (1915), and the effect of an absorbing barrier on Brownian motion was addressed by Bachelier (1900; 1901) prior to von Smoluchowski (1915; 1916). For a detailed summary of these early results the reader is referred to von Smoluchowski (1912;1916).

Jevons (1878) pointed out that the chaotic movement of microscopic particles suspended in liquids had been noted long before Brown (1827) published his careful observations; however, it should be noted that Brown (1827) was the first to emphasize its ubiquity and to exclude its explanation as a biological phenomenon. A precise definition of the Brownian motion involves a measure on the path space that was first provided by Borel (1909) and constituted the basis of the formal theory of Wiener (1921a; 1921b; 1923).

Bachelier assumed stock price dynamics with a Brownian motion without drift (resulting in a normal distribution for the stock prices), and no time-value of money. The formula provided may be used to valuate a European style call option. Later on, Kruizenga (1956) obtained the
same results as Bachelier (1900). As pointed out by Merton (1973) and Smith (1976), this approach allows negative realizations for both stock and option prices. Moreover, the option price may exceed the price of its underlying asset.

Kendall (1953), Roberts (1959), Osborne (1959; 1964) and Samuelson (1965) modified the Bachelier model (also known as the “arithmetic Brownian motion” model) assuming that the return rates, instead of the stock prices, follow a Brownian motion (also known as the “geometric Brownian motion” model or the “economic Brownian motion” model). As a result of the geometric Brownian motion the stock prices follow a log-normal distribution, instead of a normal distribution as assumed by Bachelier (1900). Sprenkle (1961; 1964) took into account risk aversion and the drift of the Brownian motion, and based upon the log-normal distribution of the stock prices, provided a new formula for the valuation of a European style call option that rules out negative option prices. Boness (1964) improved the model of Sprenkle by considering the time value of money: the present value of a call option is the discounted value provided by Sprenkle (1961) using the expected rate of return of the stock as the discount rate. Samuelson (1965) provided a rigorous review of the option valuation theory and pointed out that an option may have a different level of risk when compared with a stock, and therefore the discount rate used by Boness (1964) is incorrect. Samuelson and Merton (1969) provided a general equilibrium formula that depends on the utility function assumed for a typical investor.

The Black and Scholes (1973) model is often regarded as either the end or the beginning of the option valuation history. Using two different approaches for the valuation of European style options, they present a general equilibrium solution that is a function of “observable” variables only, making therefore the model subject to direct empirical tests. Based on the formula of Thorp and Kassouf (1967) that determines the ratio of shares of stock to options needed to construct a hedged position, and recognizing that shares and options can be combined to construct a riskless portfolio, Black and Scholes (1973) developed an analytical model that provides a no-arbitrage value for options. An alternative derivation is based upon the capital asset pricing model that provides a general method for discounting under uncertainty. Merton (1973) performed a rigorous analysis of the Black and Scholes (1973) model analyzing its assumptions. The stock price dynamics is described by a Brownian motion with drift. The manifest characteristic of the final valuation formula is the parameters it does not depend on. The option price does not depend on the expected return rate of the stock or the risk preferences of the investors. It is not assumed that the investors agree on the expected return rate of the stock. It is expected that investors may have quite different estimates for current and future returns. However, the option price depends on the risk-free interest rate and on the variance of the return rate of the stock. A detailed analysis of the post Black-Scholes models is presented by Smithson (1992).

Galai (1978) provided the correct discount rate for options, reconciling the Boness (1964) - Samuelson (1965) approach with the Black and Scholes (1973) formula. The Black and Scholes (1973) formula is identical with the Boness (1964) formula if instead of the return rate of the stock we use the risk-free interest rate. However, this risk-neutral approach may lead to confusion because it may be inferred that it can be proved that the return rate of the stock equals the risk-free interest rate (Wilmott et al. 1997).

For European style options with a single barrier, analytical solutions were provided by Merton (1973), Cox and Rubinstein (1985) and Rubinstein and Reiner (1991). Kunitomo and Ikeda (1992) provided an analytical solution for options with a double barrier but without taking into account the stock yield and the rebates corresponding to the barriers.

**DETERMINISTIC MODEL**

The deterministic model reads as follows:

\[
\frac{dS}{dt} = r S \quad \text{with} \quad S = S_0 \quad \text{at} \quad t = 0
\]

Using the substitution \( y = \ln \left( \frac{S}{S_0} \right) \) we simplify the above model:

\[
\frac{dy}{dt} = r \quad \text{with} \quad y = 0 \quad \text{at} \quad t = 0
\]

**STOCHASTIC MODEL**

Suppose that the \( r \) constant is the sum of a nominal value and of its perturbation:

\[ r = \mu + \mu' \]

The fluctuations, \( \mu' \), can be considered as a Gaussian white noise stochastic process, that is with zero expectation and the stationary autocorrelation function given by the "Dirac delta function" multiplied by a constant. This implies that the \( r \) constant can change infinitely fast. White noise is not physically realizable, because no process can change infinitely fast. Nevertheless it is often employed as a model for random physical systems. It is related to the Wiener process (Wiener 1921a; Wiener 1921b; Wiener 1923), a continuous parameter Gaussian process with zero expectation and stationary independent increments. Although the Wiener process is not differentiable, it can be shown that formally its derivative is the white noise process (Jazwinski, 1970). The ordinary differential equation for \( S \) becomes:

\[
dS = f \, dt + g \, dW
\]

where:
It is given that:

\[ f = \mu S \quad \text{and} \quad g = S \]

and \( dw \) is a Wiener process having its variance is given by:

\[ E[dw \, dw] = \sigma^2 \, dt \]

There exist two alternative interpretations of the above stochastic differential equation, the Ito and Stratonovich interpretations. The differences arise how to compute the stochastic integral \( \int g \, dw \) which appears when we integrate the stochastic differential equation. These different interpretations generally yield different solutions and there is no mathematical reason to prefer one interpretation over the other.

**Ito interpretation.** The stochastic integral in the sense of Ito (1944; 1946; 1951) is defined as follows:

\[
\int_{a}^{b} g[w(t),t] \, dw = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} g[w(t_i),t_i] \, [w(t_{i+1}) - w(t_i)]
\]

where \( a = t_0 < t_1 < t_2 \ldots < t_n = b \) is a partition of the interval \([ a , b \) with

\[
\Delta t = \max (t_{i+1} - t_i)
\]

and l.i.m. stands for "limit in the mean" (Jazwinski, 1970). When the stochastic differential equation is interpreted in the Ito sense, the rate of change of the transition probability density function \( p \) is given by the Fokker-Planck (von Smoluchowski 1906; Fokker 1914; Fokker 1917; Plank 1917) or forward Kolmogorov (1931) equation:

\[
\frac{\partial p}{\partial t} = - \frac{\partial (f \, p)}{\partial S} + \sigma^2 \frac{\partial^2 (g^2 \, p)}{2 \, \partial S^2}
\]

The initial condition selected for the Fokker-Planck equation is:

\[ p(S,0) = \delta(S_0) \]

which corresponds to the usual deterministic initial condition. The solution of the Fokker-Planck equation completely specifies the process described by the stochastic differential equation.

**Stratonovich interpretation.** The stochastic integral, in the sense of Stratonovich (1964; 1966), is defined as follows:
\[
\int_a^b g(w(t),t) \, dw = \lim_{\Delta t \to 0} \frac{1}{n-1} \sum_{i=0}^{n-1} g\left(\frac{w(t_{i+1}) + w(t_i)}{2}, t_i\right) [w(t_{i+1}) - w(t_i)]
\]

where we have adopted the same notations as for the Ito integral.

We observe that Stratonovich uses a "symmetric" definition of the stochastic integral which can be related to the Ito definition. The stochastic differential equation, in the Ito sense, corresponding to the Stratonovich interpretation of the same equation is (Jazwinski, 1970):

\[
dS = \left[ f + \frac{\sigma^2}{2} g \frac{\partial g}{\partial S} \right] dt + g \, dw
\]

Table I lists the results for our particular case. We mention that the process dynamics is governed by 2 parameters, \( \mu \) and \( \sigma \). The two interpretations give similar solutions, but \( \mu \) from the Ito case should be replaced by \( (\mu + \sigma^2/2) \) in order to obtain the Stratonovich solution. Therefore, if this model is used to fit the market data, both Ito and Stratonovich interpretations give the same prediction but using slightly different values for their \( \mu \) parameters. We have no mathematical reason to make a choice between the Ito and Stratonovich interpretations. The fact that there are two interpretations of the white noise that yield two different solutions is due to the pathological nature of the white noise and Wiener processes.

If, instead of \( S \), we investigate the dynamics of \( y \), the Ito and Stratonovich interpretations become identical. The results are presented in Table II.

The main result is that the stock price is distributed according to the lognormal law, and not to the normal one.

**PARAMETERS ESTIMATION**

**Maximum likelihood estimates.** Estimates of \( \mu \) and \( \sigma \) can be obtained from the stock prices at times \( 0, t_1, t_2, \ldots, t_N \). The recorded stock values are denoted as \( S(0) = S_0, S(t_1) = S_1, \ldots, S(t_N) = S_N \) and the time intervals between observations are \( t_1-0 = \Delta t_1, t_2-t_1 = \Delta t_2, \ldots, t_N-t_{N-1} = \Delta t_N \). The likelihood function \( L(\mu, \sigma) \) is defined as the joint probability density function for the stock price given the recorded values. We underline that the probability density function of \( S_i \) given \( S_{i-1} \) is a function of \( \Delta t_i \) but not of \( t_i \). The results obtained are listed in Tables III and IV. As usually, the Maximum Likelihood Estimate of \( \sigma^2 \) is biased, an unbiased one being given by:

\[
\sigma^2 = \frac{N}{N - 1} \sigma^2_{MLE}
\]
We have to underline that the unbiased estimate of \( \sigma \) is not the square root of the unbiased estimate of the variance, but it is given by:

\[
\sigma_{\text{Unbiased}} = \sqrt{\left( \frac{\frac{N - 1}{2}}{\Gamma\left(\frac{N}{2}\right)} \right)} \frac{\Gamma\left(\frac{N - 1}{2}\right)}{\sqrt{\sigma^2_{\text{Unbiased}}}}
\]

This result was derived independently by Pearson (1915), Deming and Birge (1934), Treloar (1935), and Holtzman (1950). Jarrett (1968) provides a detailed review.

**Linear regression approach.** The approach requires transforming the data as follows:

\[
z_i = \frac{\ln\left(\frac{S_i}{S_{i-1}}\right)}{\sqrt{\Delta_i}} \quad \text{and} \quad x_i = \sqrt{\Delta_i}
\]

Performing a linear regression without intercept using \( z \) as the dependent variable and \( x \) as the independent one, we recognize \( \mu_{\text{MLE}} \) provided by Stratonovich interpretation as an estimate for the slope parameter and \( \sigma^2 \) provided by the Stratonovich interpretation as the unbiased estimate of the error variance. According to the linear model theory, the distribution of \( \mu \) is normal, its mean given by \( \mu_{\text{MLE}} \) and its variance equal to \( \sigma^2 / t_N \). We mention that the estimate of \( \mu \) is independent of \( \sigma^2 \) and a 100\( (1-\alpha) \)% confidence interval for \( \mu \) is given by

\[
( \mu_{\text{MLE}} - t_{\alpha/2, N-1} \sqrt{\frac{\sigma^2}{t_N}} , \quad \mu_{\text{MLE}} + t_{\alpha/2, N-1} \sqrt{\frac{\sigma^2}{t_N}} )
\]

where \( t_{\alpha, N} \) is the corresponding critical value of the Student t-distribution. Moreover, the distribution of \( N \frac{\sigma^2_{\text{MLE}}}{\sigma^2} \) is chi-square with \( (N - 1) \) degrees of freedom, and a 100\( (1-\alpha) \)% confidence interval for \( \sigma^2 \) is given by:

\[
\left( \frac{N \sigma^2_{\text{MLE}}}{\chi^2_{\alpha_1, N-1}}, \frac{N \sigma^2_{\text{MLE}}}{\chi^2_{1-\alpha_2, N-1}} \right)
\]

where \( \alpha_1 + \alpha_2 = \alpha \), \( \alpha_1 > 0 \), \( \alpha_2 > 0 \). Because the distribution of \( (N - 1) \frac{\sigma^2_{\text{Unbiased}}}{\sigma^2} \) is chi-square with \( (N - 1) \) degrees of freedom, a 100\( (1-\alpha) \)% confidence interval for \( \sigma^2 \) is given by

\[
\left( \frac{(N - 1) \sigma^2_{\text{Unbiased}}}{\chi^2_{\alpha_1, N-1}}, \frac{(N - 1) \sigma^2_{\text{Unbiased}}}{\chi^2_{1-\alpha_2, N-1}} \right)
\]

**MODEL PREDICTIONS**

The expected values for stock values for both interpretations are listed in Table I. The 100\( (1-\alpha) \)% confidence interval is:
\[
(S_0 \exp[(\mu - \sigma^2/2)t - z_{\alpha/2} \sigma \sqrt{t}], S_0 \exp[(\mu - \sigma^2/2)t + z_{\alpha/2} \sigma \sqrt{t}])
\]

for the Ito interpretation, and

\[
(S_0 \exp[\mu t - z_{\alpha/2} \sigma \sqrt{t}], S_0 \exp[\mu t + z_{\alpha/2} \sigma \sqrt{t}])
\]

for the Stratonovich interpretation. Although the confidence intervals have different algebraic expressions, the numerical predictions are identical as long as the corresponding estimated values of the parameters are used. For the logarithmic transformation of the stock prices, both interpretations give the same expected value (listed in Table 2) and the same confidence interval:

\[
(\mu t - z_{\alpha/2} \sigma \sqrt{t}, \mu t + z_{\alpha/2} \sigma \sqrt{t})
\]

**GEOMETRIC BROWNIAN MOTION WITH A DOUBLE ABSORBING BARRIER**

In order to value double barrier options it may be useful to consider the trajectories of the stock prices in the presence of a double absorbing barrier (Feller 1954). Given \( L \leq S_0 \leq U \), two supplementary boundary conditions are added to the Fokker-Planck equation for \( S \):

\[
p(L,t)=0 \quad ; \quad p(U,t)=0
\]

Suppose that \( S(t) \) follows the geometric Brownian motion in the presence of a double absorbing barrier. Then the transition probability density function is:

\[
p(S,t)= \frac{1}{S \sigma \sqrt{2 \pi t}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left[-\frac{\ln(S/S_0) - (\mu^*) t - 2 n \ln(U/L)}{2 \sigma^2 t}\right] + \frac{(\mu^*) 2 n \ln(U/L)}{\sigma^2} \exp\left[-\frac{\ln(S/S_0) - (\mu^*) t - 2 n \ln(U/L) + 2 n \ln(U/L)}{2 \sigma^2 t}\right]ight. \\
\left. + \frac{(\mu^*) 2 n \ln(U/L)}{\sigma^2} \exp\left[-\frac{\ln(S/S_0) - (\mu^*) t + 2 n \ln(U/L)}{2 \sigma^2 t}\right]\right\}
\]

where:

\[
(\mu^*) = \mu - \frac{\sigma^2}{2} \quad \text{for the Ito interpretation;}
\]

\[
(\mu^*) = \mu \quad \text{for the Stratonovich interpretation}
\]
This result is an extension of the well-known formula by Lévy (1965). The solution corresponding to \( y = \ln \left( \frac{S}{S_0} \right) \) has been obtained by Bachelier (1901) and, as mentioned above, is the same for both interpretations:

\[
p(y,t) = \frac{1}{\sigma \sqrt{2 \pi t}} \sum_{n=\infty}^{\infty} \left\{ \exp\left( - \frac{(y - \mu t - 2 n \ln \left( \frac{U}{L} \right))^2}{2 \sigma^2 t} \right) + \frac{\mu 2 n \ln \left( \frac{U}{L} \right) - 2 n \ln \left( \frac{U}{S_0} \right)}{\sigma^2} \right\}\]

The probabilities of extinction due to each absorbing barrier are defined as:

\[
g_U(t) = -\frac{d}{dt} \int_0^U p(S,t) dS
\]

\[
g_L(t) = -\frac{d}{dt} \int_L^\infty p(S,t) dS
\]

while the total probability of extinction is the sum of the probabilities of extinction due to each absorbing barrier (Darling and Siegert 1953). The probability of extinction for each barrier is:

\[
g_U = \frac{1}{\sigma t \sqrt{2 \pi t}} \sum_{n=\infty}^{\infty} \left( \frac{U}{L} \right)^{2 \mu^* n} \left[ \ln \left( \frac{U}{S_0} \right) - 2 n \ln \left( \frac{U}{L} \right) \right] \cdot \exp\left( - \frac{(\ln \left( \frac{U}{S_0} \right) - (\mu^* t) \cdot \frac{U}{L} - 2 n \ln \left( \frac{U}{L} \right))^2}{2 \sigma^2 t} \right)
\]

\[
g_L = \frac{1}{\sigma t \sqrt{2 \pi t}} \sum_{n=\infty}^{\infty} \left( \frac{L}{S_0} \right)^{2 \mu^* n} \left[ \ln \left( \frac{S_0}{L} \right) + 2 n \ln \left( \frac{U}{L} \right) \right] \cdot \exp\left( - \frac{(\ln \left( \frac{L}{S_0} \right) - (\mu^* t) \cdot \frac{L}{S_0} - 2 n \ln \left( \frac{U}{L} \right))^2}{2 \sigma^2 t} \right)
\]

where \((\mu^*)\) is as specified above. The probability of extinction by hitting the upper barrier before
hitting the lower barrier is:

\[ p_{UL} = \frac{L}{S_0} \frac{2 \cdot (\mu^*) \sigma}{2 \cdot (\mu^*) \sigma^2} \left( 1 - \left( \frac{L}{U} \right) \frac{2 \cdot (\mu^*) \sigma}{2 \cdot (\mu^*) \sigma^2} \right) \]

where (\mu^*) is as specified above. This result is an extension of the formula of Darling and Siegert (1953). The cases when either \( U \to \infty \) or \( L \to 0 \) reduce to a single absorbing barrier (Bachelier 1900; Schrödinger 1915; von Smoluchowski 1915; von Smoluchowski 1916).

**RISK-NEUTRAL APPROACH**

The Black-Scholes (1973) equation for both Ito and Stratonovich interpretations is:

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - D) \cdot S \frac{\partial V}{\partial S} - r \cdot V = 0
\]

where \( V \) is the value of the option, \( r \) is the risk-free return rate, and \( D \) is the yield. This result has already been proved by Black and Scholes (1973) for the Ito interpretation. Its demonstration for the Stratonovich interpretation follows exactly the same steps.

The same result for the valuation of a European style option is obtained when:

1. The stock dynamics are described by a geometric Brownian model with the drift equal to the risk-free interest rate;
2. The discount rate is equal to the risk-free rate; and
3. The Ito interpretation is used.

This risk-neutral (or equivalent martingale) approach can be traced back to Arrow (1953). For derivatives, it was introduced by Cox and Ross (1976) and subsequently developed by Harrison and Kreps (1979), Harrison and Pliska (1981) and Kreps (1982). The results of Rubinstein and Reiner (1991) and Kunitomo and Ikeda (1992) are formulated based upon such a risk-neutral approach. A Monte-Carlo simulation designed to estimate the value of the options may be based upon this approach. However, this risk-neutral approach is valid for option valuation only, and it should not be used to simulate the stock dynamics. As pointed out by Wilmott et al. (1997), it is incorrect to conclude that the return rate of the stock equals the risk-free interest rate.

**CONCLUSIONS**

The white noise, although widely used as a model for random systems, is not physically realizable because no process can change infinitely fast. The fact that there are two
interpretations of the white noise that yield two different solutions is due to the pathological nature of the white noise and Wiener processes. There is no mathematical reason to prefer either interpretation. When more realistic correlated noise models are used, the Ito and Stratonovich interpretations become identical. As long as a model based upon the white noise is fitted to the market values, the two interpretations will provide different estimates of the parameters, but identical values concerning the predicted stock prices.

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### TABLE I. STOCHASTIC ORDINARY DIFFERENTIAL EQUATION

\[ dS = \mu S \, dt + S \, dw \]

<table>
<thead>
<tr>
<th>Fokker-Plank Equation</th>
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<tbody>
<tr>
<td><strong>Ito Interpretation</strong></td>
</tr>
<tr>
<td>[ \frac{\partial p}{\partial t} = -\mu \frac{\partial (Sp)}{\partial S} + \frac{\sigma^2}{2} \frac{\partial^2 (S^2 p)}{\partial S^2} ]</td>
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<tr>
<td><strong>Stratonovich Interpretation</strong></td>
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<tr>
<td>[ \frac{\partial p}{\partial t} = -(\mu + \frac{\sigma^2}{2}) \frac{\partial (Sp)}{\partial S} + \frac{\sigma^2}{2} \frac{\partial^2 (S^2 p)}{\partial S^2} ]</td>
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<tr>
<th>Transition probability density function</th>
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<tr>
<td><strong>Ito Interpretation</strong></td>
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<tr>
<td>[ p(S,t) = \exp\left{ -\frac{[\ln(S/S_0) - (\mu - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t}\right} \frac{S}{2\sqrt{2\pi} \sigma^2 t} ]</td>
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<td>[ p(S,t) = \exp\left{ -\frac{[\ln(S/S_0) - \mu t]^2}{2\sigma^2 t}\right} \frac{S}{S\sqrt{2\pi \sigma^2 t}} ]</td>
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<th>Moments</th>
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<tr>
<td><strong>Ito Interpretation</strong></td>
</tr>
<tr>
<td>[ E[S^n] = S_0^n \exp[n \mu t + n(n-1)(\sigma^2/2)t] ]</td>
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<th>Average</th>
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<tr>
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<tr>
<td>[ E[S] = S_0 \exp(\mu t) ]</td>
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<tr>
<td><strong>Stratonovich Interpretation</strong></td>
</tr>
<tr>
<td>[ E[S] = S_0 \exp(\mu + \frac{\sigma^2}{2}t) ]</td>
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<tr>
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<tr>
<td><strong>Ito Interpretation</strong></td>
</tr>
<tr>
<td>[ Var[S] = S_0^2 \exp(2\mu t) [\exp(\sigma^2 t) - 1] ]</td>
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<tr>
<td><strong>Stratonovich Interpretation</strong></td>
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<td>[ Var[S] = S_0^2 \exp(2(\mu + \sigma^2/2)t) [\exp(\sigma^2 t) - 1] ]</td>
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<th>Median</th>
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<tr>
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<tr>
<td>[ Median[S] = S_0 \exp[(\mu - \frac{\sigma^2}{2})t] ]</td>
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<tr>
<td><strong>Stratonovich Interpretation</strong></td>
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<tr>
<td><strong>TABLE II. STOCHASTIC ORDINARY DIFFERENTIAL EQUATION</strong> dys = μ dt + dw</td>
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<td>---------------------------------------------------------------</td>
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<tr>
<td><strong>Ito or Stratonovich Interpretation</strong></td>
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<tr>
<td><strong>Fokker-Plank Equation</strong></td>
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</tr>
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<td><strong>Transition probability density function</strong></td>
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<tr>
<td><strong>Moments</strong></td>
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<td>[ E[y^n] = \sum_{m=0}^{[n/2]} \binom{n}{2m} (2\sigma^2 t)^m (\mu t)^{n-2m} \frac{\Gamma(m+1/2)}{\sqrt{\pi}} ]</td>
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<td>[ E[y] = \mu t ]</td>
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</tbody>
</table>
### Table III. Parameters Identification for the Stochastic Ordinary Differential Equation

\( dS = \mu S \, dt + S \, dw \)

<table>
<thead>
<tr>
<th>Ito Interpretation</th>
<th>Stratonovich Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ L(\mu, \sigma) = \prod_{i=1}^{N} \exp\left{ \frac{- \left[ \ln \left( \frac{S_i}{S_{i-1}} \right) - \left( \frac{\mu - \sigma^2}{2} \right) \Delta t_i \right]^2}{2 \sigma^2 \Delta t_i} \right} \frac{S_i \sqrt{2\pi \sigma^2 \Delta t_i}}{S_i \sqrt{2\pi \sigma^2 \Delta t_i}} ]</td>
<td>[ L(\mu, \sigma) = \prod_{i=1}^{N} \exp\left{ \frac{- \left[ \ln \left( \frac{S_i}{S_{i-1}} \right) - \mu \Delta t_i \right]^2}{2 \sigma^2 \Delta t_i} \right} \frac{S_i \sqrt{2\pi \sigma^2 \Delta t_i}}{S_i \sqrt{2\pi \sigma^2 \Delta t_i}} ]</td>
</tr>
<tr>
<td>[ \sigma^2_{MLE} = \frac{1}{N} \sum_{i=1}^{N} \Delta t_i \left[ \frac{1}{\Delta t_i} \ln \left( \frac{S_i}{S_{i-1}} \right) - \frac{1}{t_N} \ln \left( \frac{S_N}{S_0} \right) \right]^2 ]</td>
<td>[ \sigma^2_{MLE} = \frac{1}{N} \sum_{i=1}^{N} \Delta t_i \left[ \frac{1}{\Delta t_i} \ln \left( \frac{S_i}{S_{i-1}} \right) - \frac{1}{t_N} \ln \left( \frac{S_N}{S_0} \right) \right]^2 ]</td>
</tr>
<tr>
<td>[ \mu = \frac{1}{t_N} \ln \left( \frac{S_N}{S_0} \right) + \frac{\sigma^2}{2} ]</td>
<td>[ \mu = \frac{1}{t_N} \ln \left( \frac{S_N}{S_0} \right) ]</td>
</tr>
</tbody>
</table>
TABLE IV. PARAMETERS IDENTIFICATION FOR THE STOCHASTIC ORDINARY
DIFFERENTIAL EQUATION
\[
dy = \mu \, dt + \, dw
\]

<table>
<thead>
<tr>
<th>Ito or Stratonovich Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ L(\mu, \sigma) = \prod_{i=1}^{N} \exp\left{ -\frac{\left[ y_i - y_{i-1} - \mu \Delta t_i \right]^2}{2 \sigma^2 \Delta t_i} \right} ]</td>
</tr>
<tr>
<td>[ \sigma_{MLE}^2 = \frac{1}{N} \sum_{i=1}^{N} \Delta t_i \left[ \frac{1}{\Delta t_i} \left( y_i - y_{i-1} \right) - \frac{1}{t_N} \sum_{i=1}^{N} y_i \right]^2 ]</td>
</tr>
<tr>
<td>[ \mu = \frac{1}{t_N} \sum_{i=1}^{N} y_i ]</td>
</tr>
</tbody>
</table>