STOCHASTIC MODELING OF STOCK PRICES

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ABSTRACT

The geometric Brownian motion model is widely used to explain the stock price time series. The following sections summarize its main features. The stochastic model may be viewed as an extension of the usual deterministic model for which the rate of return is viewed as a constant value subjected to perturbations. We present both the Ito and Stratonovich interpretations of the resulting stochastic differential equation. The parameters estimation and model predictions could be done using either interpretation; however, the same interpretation must be used for both steps (i.e., parameters estimation and model predictions).

INTRODUCTION

Bachelier (1900) seems to be the first to have provided an analytical valuation for stock options. His work is rather remarkable because by addressing the problem of option pricing, Bachelier (1900) derived most of the theory of diffusion processes. The mathematical theory of Brownian motion has been formulated by Bachelier (1900) five years before Einstein's classic paper (Einstein 1905). Bachelier (1900) has formulated *avant la lettre* the Chapman-Kolmogorov equation (von Smoluchowski 1906; Chapman 1916; Chapman 1917; Kolmogorov 1931), called today the Chapman-Kolmogorov-Smoluchowski-Bachelier equation (Brown et al. 1995), and the Fokker-Plank or Kolmogorov equation (von Smoluchowski 1906; Fokker 1914; Fokker 1917; Plank 1917; Kolmogorov 1931). Moreover, the first-passage distribution function for the drift-free case was provided by Bachelier (1900) before Schrödinger (1915), and the effect of an absorbing barrier on Brownian motion was addressed by Bachelier (1900; 1901) prior to von Smoluchowski (1915; 1916). For a detailed summary of these early results the reader is referred to von Smoluchowski (1912;1916).

Jevons (1878) pointed out that the chaotic movement of microscopic particles suspended in liquids had been noted long before Brown (1827) published his careful observations; however, it should be noted that Brown (1827) was the first to emphasize its ubiquity and to exclude its explanation as a biological phenomenon. A precise definition of the Brownian motion involves a measure on the path space that was first provided by Borel (1909) and constituted the basis of the formal theory of Wiener (1921a; 1921b; 1923).

Bachelier assumed stock price dynamics with a Brownian motion without drift (resulting in a normal distribution for the stock prices), and no time-value of money. The formula provided may be used to valuate a European style call option. Later on, Kruizenga (1956) obtained the same results as Bachelier (1900). As pointed out by Merton (1973) and Smith (1976), this approach allows negative realizations for both stock and option prices. Moreover, the option price may exceed the price of its underlying asset.

Kendall (1953), Roberts (1959), Osborne (1959; 1964) and Samuelson (1965) modified the Bachelier model (also known as the "arithmetic Brownian motion" model) assuming that the return rates, instead of the stock prices, follow a Brownian motion (also known as the "geometric Brownian motion" model or the "economic Brownian motion" model). As a result of the geometric Brownian motion the stock prices follow a log-normal distribution, instead of a normal distribution as assumed by Bachelier (1900). Sprenkle (1961; 1964) took into account risk aversion and the drift of the Brownian motion, and based upon the log-normal distribution of the stock prices, provided a new formula for the valuation of a European style call option that rules out negative option prices. Boness (1964) improved the model of Sprenkle by considering the time value of money: the present value of a call option is the discounted value provided by Sprenkle (1961) using the expected rate of return of the stock as the discount rate. Samuelson (1965) provided a rigorous review of the option valuation theory and pointed out that an option may have a different level of risk when compared with a stock, and therefore the discount rate used by Boness (1964) is incorrect. Samuelson and Merton (1969) provided a general equilibrium formula that depends on the utility function assumed for a typical investor.

The Black and Scholes (1973) model is often regarded as either the end or the beginning of the option valuation history. Using two different approaches for the valuation of European style options, they present a general equilibrium solution that is a function of "observable" variables only, making therefore the model subject to direct empirical tests. Based on the formula of Thorp and Kassouf (1967) that determines the ratio of shares of stock to options needed to construct a hedged position, and recognizing that shares and options can be combined to construct a riskless portfolio, Black and Scholes (1973) developed an analytical model that provides a no-arbitrage value for options. An alternative derivation is based upon the capital asset pricing model that provides a general method for discounting under uncertainty. Merton (1973) performed a rigorous analysis of the Black and Scholes (1973) model analyzing its assumptions. The stock price dynamics is described by a Brownian motion with drift. The manifest characteristic of the final valuation formula is the parameters it does not depend on. The option price does not depend on the expected return rate of the stock or the risk preferences of the investors. It is not assumed that the investors agree on the expected return rate of the stock. It is expected that investors may have quite different estimates for current and future returns. However, the option price depends on the risk-free interest rate and on the variance of the return rate of the stock. A detailed analysis of the post Black-Scholes models is presented by Smithson (1992).

Galai (1978) provided the correct discount rate for options, reconciling the Boness (1964) - Samuelson (1965) approach with the Black and Scholes (1973) formula. The Black and Scholes (1973) formula is identical with the Boness (1964) formula if instead of the return rate of the stock we use the risk-free interest rate. However, this risk-neutral approach may lead to confusion because it may be inferred that it can be proved that the return rate of the stock equals the risk-free interest rate (Wilmott et al. 1997).

Barrier options have become increasingly popular over the last years (Johnson and Stulz 1987; Boyle and Turnbull 1989; Benson and Daniel 1991; Hudson 1991; Hudson 1995; Derman and Kani 1993; Ravindran 1993; Bowie and Carr 1994; Hart and Ross 1994; Heynen and Kat 1994a; Heynen and Kat 1994b; Rich 1994; Schnabel and Wei 1994; Broadie and Detemple 1995; Hull and White 1995; Jarrow and Turnbull 1995; Zhang 1995; Gerber and Shiu 1996). Practitioners who trade these instruments rely heavily on the numerical solutions provided by McConnell and Schwartz (1986), Trippi and Chance (1993), Boyle and Lau (1994), Derman et al. (1994), Kat and Verdonk (1995), Ritchken (1995), Geman and Yor (1996), Dewynne and Willmott (1997). For European style options with a single barrier, analytical solutions were provided by Merton (1973), Cox and Rubinstein (1985) and Rubinstein and Reiner (1991). Kunitomo and Ikeda (1992) provided an analytical solution for options with a double barrier but without taking into account the stock yield and the rebates corresponding to the barriers.

DETERMINISTIC MODEL

The deterministic model reads as follows:

$$\frac{dS}{dt} = r\,S \qquad with \qquad S = S_0 \quad at \quad t = 0$$

Using the substitution $\mathbf{y = \ln (S/S_0)}$ we simplify the above model:

$$\frac{dy}{dt} = r \qquad with \qquad y = 0 \quad at \quad t = 0$$

STOCHASIC MODEL

Suppose that the **r** constant is the sum of a nominal value and of its perturbation:

$$r = \mu + \mu'$$

The fluctuations, $\boldsymbol{\mu}'$, can be considered as a Gaussian white noise stochastic process, that is with zero expectation and the stationary autocorrelation function given by the "Dirac delta function" multiplied by a constant. This implies that the **r** constant can change infinitely fast. White noise is not physically realizable, because no process can change infinitely fast. Nevertheless it is often employed as a model for random physical systems. It is related to the Wiener process (Wiener 1921a; Wiener 1921b; Wiener 1923), a continuous parameter Gaussian process with zero expectation and stationary independent increments. Although the Wiener process is not differentiable, it can be shown that formally its derivative is the white noise process (Jazwinski, 1970). The ordinary differential equation for **S** becomes:

$$dS = f\ dt + g\ dW$$

where:

$$f = \mu S \qquad and \qquad g = S$$

and **dw** is a Wiener process having its variance is given by:

$$E[dw\, dw] = \sigma^2\, dt$$

There exist two alternative interpretations of the above stochastic differential equation, the Ito and Stratonovich interpretations. The differences arise how to compute the stochastic integral $\int$**gdw** which appears when we integrate the stochastic differential equation. These different interpretations generally yield different solutions and there is no mathematical reason to prefer one interpretation over the other.

Ito interpretation. The stochastic integral in the sense of Ito (1944; 1946; 1951) is defined as follows:

$$\int_a^b g[w(t),t]\ dw = \frac{l.i.m.}{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} g[w(t_i),t_i]\ [w(t_{i+1}) - w(t_i)]$$

where $a = t_0 < t_1 < t_2 \ \ldots < t_n = b$ is a partition of the interval $[\ a\ ,\ b\ ]$ with

$$\Delta t = \max_i\ (t_{i+1} - t_i)$$

and l.i.m. stands for "limit in the mean" (Jazwinski, 1970). When the stochastic differential equation is interpreted in the Ito sense, the rate of change of the transition probability density function p is given by the Fokker-Planck (von Smoluchowski 1906; Fokker 1914; Fokker 1917; Plank 1917) or forward Kolmogorov (1931) equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial (f\ p)}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 (g^2\ p)}{\partial S^2}$$

The initial condition selected for the Fokker-Planck equation is:

$$p(S,0) = \delta(S_0)$$

which corresponds to the usual deterministic initial condition. The solution of the Fokker-Planck equation completely specifies the process described by the stochastic differential equation.

Stratonovich interpretation. The stochastic integral, in the sense of Stratonovich (1964; 1966), is defined as follows:

$$\int_a^b g[w(t),t]\,dw = \underset{\Delta t \to 0}{l.i.m.} \sum_{i=0}^{n-1} g[\frac{w(t_{i+1}) + w(t_i)}{2}, t_i]\,[w(t_{i+1}) - w(t_i)]$$

where we have adopted the same notations as for the Ito integral.

We observe that Stratonovich uses a "symmetric" definition of the stochastic integral which can be related to the Ito definition. The stochastic differential equation, in the Ito sense, corresponding to the Stratonovich interpretation of the same equation is (Jazwinski, 1970):

$$dS = [f + \frac{\sigma^2}{2}\, g\, \frac{\partial g}{\partial S}]\,dt + g\,dw$$

Table I lists the results for our particular case. We mention that the process dynamics is governed by 2 parameters, μ and σ. The two interpretations give similar solutions, but μ from the Ito case should be replaced by $(\mu + \sigma^2/2)$ in order to obtain the Stratonovich solution. Therefore, if this model is used to fit the market data, both Ito and Stratonovich interpretations give the same prediction but using slightly different values for their μ parameters. We have no mathematical reason to make a choice between the Ito and Stratonovich interpretations. The fact that there are two interpretations of the white noise that yield two different solutions is due to the pathological nature of the white noise and Wiener processes.

If, instead of **S**, we investigate the dynamics of **y**, the Ito and Stratonovich interpretations become identical. The results are presented in Table II.

The main result is that the stock price is distributed according to the lognormal law, and not to the normal one.

PARAMETERS ESTIMATION

Maximum likelihood estimates. Estimates of μ and σ can be obtained from the stock prices at times $o, t_1, t_2, \ldots, t_N$. The recorded stock values are denoted as $S(0) = S_0, S(t_1)= S_1, ..., S(t_N) = S_N$ and the time intervals between observations are $t_1\text{-}0 = \Delta t_1, t_2\text{-}t_1 = \Delta t_2, ..., t_N\text{-}t_{N-1} = \Delta t_N$. The likelihood function $L(\mu,\sigma)$ is defined as the joint probability density function for the stock price given the recorded values. We underline that the probability density function of S_i given S_{i-1} is a function of Δt_i but not of t_i. The results obtained are listed in Tables III and IV. As usually, the Maximum Likelihood Estimate of σ^2 is biased, an unbiased one being given by:

$$\sigma^2 = \frac{N}{N-1}\,\sigma^2_{MLE}$$

We have to underline that the unbiased estimate of **σ** is not the square root of the unbiased estimate of the variance, but it is given by:

$$\sigma_{Unbiased} = \sqrt{(\frac{N-1}{2})}\frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N}{2})}\sqrt{(\sigma^2_{Unbiased})}$$

This result was derived independently by Pearson (1915), Deming and Birge (1934), Treloar (1935), and Holtzman (1950). Jarrett (1968) provides a detailed review.

Linear regression approach. The approach requires transforming the data as follows:

$$z_i = \frac{\ln(\frac{S_i}{S_{i-1}})}{\sqrt{\Delta_i}} \qquad and \qquad x_i = \sqrt{\Delta_i}$$

Performing a linear regression without intercept using z as the dependent variable and x as the independent one, we recognize $\boldsymbol{\mu}_{\mathbf{MLE}}$ provided by Stratonovich interpretation as an estimate for the slope parameter and $\boldsymbol{\sigma}^2$ provided by the Stratonovich interpretation as the unbiased estimate of the error variance. According to the linear model theory, the distribution of **μ** is normal, its mean given by $\boldsymbol{\mu}_{\mathbf{MLE}}$ and its variance equal to $\boldsymbol{\sigma}^2/\mathbf{t_N}$. We mention that the estimate of **μ** is independent of $\boldsymbol{\sigma}^2$ and a 100(1-α)% confidence interval for **μ** is given by

$$(\mu_{MLE} - t_{\alpha/2,\, N-1}\sqrt{\sigma^2/t_N},\ \mu_{MLE} + t_{\alpha/2,\, N-1}\sqrt{\sigma^2/t_N})$$

where $\mathbf{t_{\alpha,N}}$ is the corresponding critical value of the Student t-distribution. Moreover, the distribution of $\mathbf{N}\ \boldsymbol{\sigma}^2_{\mathbf{MLE}}/\boldsymbol{\sigma}^2$ is chi-square with **(N - 1)** degrees of freedom, and a 100(1-α)% confidence interval for $\boldsymbol{\sigma}^2$ is given by:

$$(N\ \sigma^2_{MLE}/\chi^2_{\alpha_1, N-1},\ N\ \sigma^2_{MLE}/\chi^2_{1-\alpha_2, N-1})$$

where $\boldsymbol{\alpha_1 + \alpha_2 = \alpha}$, $\boldsymbol{\alpha_1 > 0, \alpha_2 > 0}$. Because the distribution of $\mathbf{(N-1)}\ \boldsymbol{\sigma}^2_{\mathbf{Unbiased}}/\boldsymbol{\sigma}^2$ is chi-square with **(N - 1)** degrees of freedom, a 100(1-α)% confidence interval for $\boldsymbol{\sigma}^2$ is given by

$$((N-1)\sigma^2_{Unbiased}/\chi^2_{\alpha_1,N-1},\ (N-1)\sigma^2_{Unbiased}/\chi^2_{1-\alpha_2,N-1})$$

MODEL PREDICTIONS

The expected values for stock values for both interpretations are listed in Table I. The 100(1-α)% confidence interval is:

$$(S_0 \exp[(\mu - \sigma^2/2)t - z_{\alpha/2}\,\sigma\sqrt{t}\,] \,, S_0 \exp[(\mu - \sigma^2/2)t + z_{\alpha/2}\,\sigma\sqrt{t}\,])$$

for the Ito interpretation, and

$$(S_0 \exp[\mu\, t - z_{\alpha/2}\,\sigma\sqrt{t}\,] \,, S_0 \exp[\mu\, t + z_{\alpha/2}\,\sigma\sqrt{t}\,])$$

for the Stratonovich interpretation. Although the confidence intervals have different algebraic expressions, the numerical predictions are identical as long as the corresponding estimated values of the parameters are used. For the logarithmic transformation of the stock prices, both interpretations give the same expected value (listed in Table 2) and the same confidence interval:

$$(\mu\, t - z_{\alpha/2}\,\sigma\sqrt{t} \,, \mu\, t + z_{\alpha/2}\,\sigma\sqrt{t}\,)$$

GEOMETRIC BROWNIAN MOTION WITH A DOUBLE ABSORBING BARRIER

In order to value double barrier options it may be useful to consider the trajectories of the stock prices in the presence of a double absorbing barrier (Feller 1954). Given $\mathbf{L \le S_0 \le U}$, two supplementary boundary conditions are added to the Fokker-Planck equation for **S**:

$$p(L,t) = 0 \quad ; \quad p(U,t) = 0$$

Suppose that **S(t)** follows the geometric Brownian motion in the presence of a double absorbing barrier. Then the transition probability density function is:

$$\begin{aligned} p(S,t) = \frac{1}{S\,\sigma\sqrt{2\pi t}} \sum_{n=-\infty}^{n=\infty} \Big\{ \exp\Big\{ -\frac{[\ln(\frac{S}{S_0}) - (\mu^*)t - 2n\ln(\frac{U}{L})]^2}{2\sigma^2 t} + \\ + \frac{(\mu^*)\,2n\ln(\frac{U}{L})}{\sigma^2} \Big\} - \exp\Big\{ -\frac{[\ln(\frac{S}{S_0}) - (\mu^*)t - 2\ln(\frac{U}{S_0}) + 2n\ln(\frac{U}{L})]^2}{2\sigma^2 t} + \\ + \frac{(\mu^*)[2\ln(\frac{U}{S_0}) - 2n\ln(\frac{U}{L})}{\sigma^2} \Big\}\Big\} \end{aligned}$$

where:

$$(\mu^*) = \mu - \frac{\sigma^2}{2} \quad \text{for the Ito interpretation;}$$

$$(\mu^*) = \mu \qquad \text{for the Stratonovich interpretation}$$

This result is an extension of the well-known formula by Lévy (1965). The solution corresponding to **y = ln (S/S$_0$)** has been obtained by Bachelier (1901) and, as mentioned above, is the same for both interpretations:

$$p(y,t) = \frac{1}{\sigma\sqrt{2\pi t}} \sum_{n=-\infty}^{n=\infty} \{ \exp\{ - \frac{[y - \mu t - 2n \ln(\frac{U}{L})]^2}{2\sigma^2 t} + \frac{\mu 2n \ln(\frac{U}{L})}{\sigma^2} \} - \exp\{ - \frac{[y - \mu t - 2\ln(\frac{U}{S_0}) + 2n\ln(\frac{U}{L})]^2}{2\sigma^2 t} + \frac{\mu [2\ln(\frac{U}{S_0}) - 2n\ln(\frac{U}{L})}{\sigma^2} \}\}$$

The probabilities of extinction due to each absorbing barrier are defined as:

$$g_U(t) = -\frac{d}{dt}\int_0^U p(S,t)\,dS$$

$$g_L(t) = -\frac{d}{dt}\int_L^\infty p(S,t)\,dS$$

while the total probability of extinction is the sum of the probabilities of extinction due to each absorbing barrier (Darling and Siegert 1953). The probability of extinction for each barrier is:

$$g_U = \frac{1}{\sigma t\sqrt{2\pi t}} \sum_{n=-\infty}^{n=\infty} (\frac{U}{L})^{\frac{2(\mu^*)n}{\sigma^2}} [\ln(\frac{U}{S_0}) - 2n\ln(\frac{U}{L})] \cdot \exp\{ - \frac{[\ln(\frac{U}{S_0}) - (\mu^*)t - 2n\ln(\frac{U}{L})]^2}{2\sigma^2 t} \}$$

$$g_L = \frac{1}{\sigma t\sqrt{2\pi t}} \sum_{n=-\infty}^{n=\infty} (\frac{U}{L})^{\frac{2(\mu^*)n}{\sigma^2}} [\ln(\frac{S_0}{L}) + 2n\ln(\frac{U}{L})] \cdot \exp\{ - \frac{[\ln(\frac{L}{S_0}) - (\mu^*)t - 2n\ln(\frac{U}{L})]^2}{2\sigma^2 t} \}$$

where (μ^*) is as specified above. The probability of extinction by hitting the upper barrier before hitting the lower barrier is:

$$p_{UL} = \frac{1 - \left(\frac{L}{S_0}\right)^{\frac{2 \cdot (\mu^*)}{\sigma^2}}}{1 - \left(\frac{L}{U}\right)^{\frac{2 \cdot (\mu^*)}{\sigma^2}}}$$

where (μ^*) is as specified above. This result is an extension of the formula of Darling and Siegert (1953). The cases when either $\mathbf{U} \to \infty$ or $\mathbf{L} \to \mathbf{0}$ reduce to a single absorbing barrier (Bachelier 1900; Schrödinger 1915; von Smoluchowski 1915; von Smoluchowski 1916).

RISK-NEUTRAL APPROACH

The Black-Scholes (1973) equation for both Ito and Stratonovich interpretations is:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \cdot \frac{\partial^2 V}{\partial S^2} + (r - D) \cdot S \cdot \frac{\partial V}{\partial S} - r \cdot V = 0$$

where **V** is the value of the option, **r** is the risk-free return rate, and **D** is the yield. This result has already been proved by Black and Scholes (1973) for the Ito interpretation. Its demonstration for the Stratonovich interpretation follows exactly the same steps.

The same result for the valuation of a European style option is obtained when:

1. The stock dynamics are described by a geometric Brownian model with the drift equal to the risk-free interest rate;

2. The discount rate is equal to the risk-free rate; and

3. The Ito interpretation is used.

This risk-neutral (or equivalent martingale) approach can be traced back to Arrow (1953). For derivatives, it was introduced by Cox and Ross (1976) and subsequently developed by Harrison and Kreps (1979), Harrison and Pliska (1981) and Kreps (1982). The results of Rubinstein and Reiner (1991) and Kunitomo and Ikeda (1992) are formulated based upon such a risk-neutral approach. A Monte-Carlo simulation designed to estimate the value of the options may be based upon this approach. However, this risk-neutral approach is valid for option valuation only, and it should not be used to simulate the stock dynamics. As pointed out by Wilmott et al. (1997), it is incorrect to conclude that the return rate of the stock equals the risk-free interest rate.

CONCLUSIONS

The white noise, although widely used as a model for random systems, is not physically realizable because no process can change infinitely fast. The fact that there are two interpretations of the white noise that yield two different solutions is due to the pathological nature of the white noise and Wiener processes. There is no mathematical reason to prefer either interpretation. When more realistic correlated noise models are used, the Ito and Stratonovich interpretations become identical. As long as a model based upon the white noise is fitted to the market values, the two interpretations will provide different estimates of the parameters, but identical values concerning the predicted stock prices.

REFERENCES

Arrow, K. J. "Le rôle des valeurs boursières pour la répartition meilleure des risques," *Econométrie Colloq. Internat. Centre National de la Recherche Scientifique* **40**, 41– 47;1953. (English translation in Review of Economic Studies **31**, 91 – 96;1964.)

Bachelier, L. "Théorie de la speculation", *Annales Scientifiques de L'École Normale Supérieure* **17**, 21 – 86;1900. (English translation by A. J. Boness in Cootner, P.H. (Editor): "The random character of stock market prices." Pages 17 – 75. Cambridge, MA: MIT Press; 1964)

Bachelier, L. "Théorie mathématique du jeu," *Annales de L'École Normale Supèrieure*, **18**, 143-210; 1901 (as cited in Lévy (1965)).

Benson, R.; Daniel, N. "Up, over and out," *Risk* **4** (June), 17 – 19; 1991.

Black, F.; Scholes, M. "The pricing of options and corporate liabilities," *Journal of Political Economy*, **81** (May - June), 637 – 659; 1973.

Boness, A. J. "Elements of a theory of stock-option value," *The Journal of Political Economy*, **72** , 163 – 175; 1964.

Borel, E. "Les probabilités dénombrables et leurs applications arithmétiques," *Rendiconti del Circolo Matematico di Palermo*, **27**, 247-271; 1909.

Bowie, J.; Carr, P. "Static simplicity," *Risk* **7** (August), 44 – 49; 1994.

Boyle, P. P.; Lau, S. H. "Bumping up against the barrier with the binomial method," *Journal of Derivatives* **1**, 6 – 14; 1994.

Boyle, P. P.; Turnbull, S. M. "Pricing and hedging capped options," *Journal of Futures Markets* **9** (January), 41-54; 1989.

Broadie, M.; Detemple, J. "American capped call options and dividend-paying assets," *Review of Financial Studies* **8** (Spring), 161 – 191; 1995.

Brown, L. M.; Pais, A.; Sir Brian Pippard (editors). "Twentieth Century Physics." Vol. I. Page 604. New York, NY: American Institute of Physics Press; 1995.

Brown, R. “A brief account of microscopical observations made in the months of June, July and August 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies,” *Edinburgh New Philosophical Journal* **5**, 358-371; 1828.

Chapman, S. “Law of distribution of molecular velocities, and theory of viscosity and thermal conduction in a non-uniform simple monoatomic gas,” *Philosophical Transactions of the Royal Society of London*, Series **A 216**, 279-348; 1916.

Chapman, S. “The kinetic theory of a composite monoatomic gas,” *Philosophical Transactions of the Royal Society of London*, Series **A 217**, 115-197; 1917.

Cox, J.; Ross, S. “The valuation of options for alternative stochastic processes,” *Journal of Financial Economics* **3**, 145 – 166; 1976.

Cox, J. C.; Rubinstein, M. *Option markets*. Englewood Cliffs, NJ: Prentice Hall; 1985.

Darling, D. A.; Siegert, A. J. F. “The first passage problem for a continuous Markov process,” *The Annals of Mathematical Statistics*, **24**, 624-639; 1953.

Deming, W. E.; Birge, R. T. “On the statistical theory of errors,” *Reviews of Modern Physics* **6**, 119 – 161; 1934.

Derman, E.; Ergener, D.; Kani, I. “Forever hedged,” *Risk* **7** (September), 139 – 145; 1994.

Derman, E.; Kani, I. “The ins and outs of barrier options,” Goldman Sachs Quantitative Strategies Research Notes; 1993.

Dewynne, J. N.; Wilmott, P. “Exotic options: Mathematical models and computation,” in *Frontiers in derivatives*, A. Konishi and R. E. Dattatreya (Eds.). Chicago, IL: Irwin Professional Publishing; 1997.

Einstein, A. “Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendiernten Teilchen,” *Annalen der Physik*, **17**, 549 – 560; 1905.

Feller, W. “Diffusion processes in one dimension,” *Transactions of the American Mathematical Society*, **77**, 1-31; 1954.

Fokker, A. D. “Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld,” *Annalen der Physik*, **43**, 810-820; 1914.

Fokker, A.D. “Sur les mouvements browniens dans le champ du rayonnement noir,” *Archives Néerlandaises des Sciences Exactes et Naturelles*, **4**, 379-401; 1917.

Galai, D. "On the Boness and Black-Scholes models for valuation of call options," *Journal of Financial and Quantitative Analysis*, **13** (March), 15 – 27; 1978.

Geman, H.; and Yor, M. "Pricing and hedging double-barrier options: A probabilistic approach," *Mathematical Finance* **6**, 365 – 378; 1996.

Gerber, H. U.; Shiu, E. S. W. "Martingale approach to pricing perpetual American options on two stocks," *Mathematical Finance* **6**, 303 – 322; 1996.

Harrison, J. M.; Kreps, D. "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory*, **20**, 381 – 408; 1979.

Harrison, J. M.; Pliska, S. R. "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and Their Applications* **11**, 215 – 260; 1981.

Hart, I.; Ross, M. "Striking continuity," *Risk* **7** (June), 51 – 56; 1994.

Heynen, R. C.; Kat, H. M. "Crossing barriers," *Risk* **7** (June), 46 – 51; 1994a.

Heynen, R. C.; Kat, H. M. "Partial barrier options," *Journal of Financial Engineering* **3**, 253 – 274; 1994b.

Holtzman, W. H. "The unbiased estimate of the population variance and standard deviation," *American Journal of Psychology*, **63**, 615 – 617; 1950.

Hudson, M. "The value of going out," *Risk*, **4** (March), 29 – 33; 1991.

Hudson, M. "The value of going out (correction)," *Risk* **8** (March), 18; 1995.

Hull, J. C.; White, A. "The impact of default risk on options and other derivative securities," *Journal of Banking and Finance*, **19**, 299 – 322; 1995.

Ito, K. "Stochastic integral," *Proceedings of the Imperial Academy Tokyo,* **20**, 519-524; 1944.

Ito, K. "On a stochastic integral equation," *Proceedings of the Japan Academy*, **22**, 32-35; 1946.

Ito, K. "On stochastic differential equations," *Memoirs American Mathematical Society*, 4; 1951.

Jarrett, R. F. "A minor exercise in history," *The American Statistician*, **22** (June), 25 – 26; 1968.

Jarrow, R. A.; Turnbull, S. M. "Pricing derivatives on financial securities subject to credit risk," *Journal of Finance*, **50**, 53 – 85; 1995.

Jazwinski, A. H. *Stochastic processes and filtering theory*. New York, NY: Academic Press; 1970.

Jevons, W. S. "On the movement of microscopic particles suspended in liquids," *Quarterly Journal of Science (London)*, April 1878.

Johnson, H.; Stulz, R. "The pricing of options with default risk," *Journal of Finance*, **42**, 267 – 280; 1987.

Kat, H.; Verdonk, L. "Tree surgery," *Risk* **8** (February), 53 – 56; 1995.

Kendall, M. S. "The analysis of economic time series - Part I: Prices," *Journal of the Royal Statistical Society*, Series **A 66**, 11-25; 1953.

Kreps, D. "Multiperiod securities and the efficient allocation of risk: A comment on the Black-Scholes option pricing model," in *The Economics of uncertainty and information*, J. J. McCall (Ed). Chicago, IL: University of Chicago; 1982.

Kruizenga, R. J. "Put and call options: A theoretical and market analysis," Ph.D. Thesis, Cambridge, MA: MIT; 1956.

Kolmogorov, A. "Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung," *Mathematische Annalen*, **104**, 415-458; 1931.

Kunitomo, N.; Ikeda, M. "Pricing options with curved boundaries," *Mathematical Finance* **2**, 275 – 298; 1992.

Lévy, P. *Processus stochastiques et mouvement brownien*. Paris: Gauthier-Villars; 1965.

McConnell, J.; Schwartz, E. S. "LYON taming," *Journal of Finance,* **41**, 561 – 577; 1986.

Merton, R. C. "The theory of rational option pricing," *The Bell Journal of Economics and Management Science*, **4** (Spring), 141 – 183; 1973.

Osborne, M. F. M. "Brownian motion in the stock market," *Operations Research*, **7**, 145 – 173; 1959.

Osborne, M. F. M. "Brownian motion in the stock market," in *The random character of stock market prices*, P. H. Cootner (Editor). Pages 100 – 128. Cambridge, MA: MIT Press; 1964.

Pearson, K. "On the distribution of standard deviations of small samples. Appendix I to papers by Student and R.A. Fisher," *Biometrika* **10**, 522 – 529; 1915.

Planck, M. "Über einen Satz der Statistischen Dynamik und seine Erweiterung in der Quantentheorie," *Sitzungsberichte der Preuss. Akademie der Wissenschaften in Berlin*, **24**, 324-341;1917.

Ravindran, K. “Option pricing: An offspring of the secretary problem?,” *Mathematica Japonica*, **38** , 905 – 912; 1993.

Rich, D. R. “The mathematical foundations of barrier options,” *Advances in Futures and Options Research*, **7**, 267 – 311; 1994.

Ritchken, P. “On pricing barrier options,” *Journal of Derivatives* **3**, 19 – 28; 1995.

Roberts, H. V. “Stock market patterns and financial analysis: Methodological suggestions,” *Journal of Finance* **14**, 1-10; 1959.

Rubinstein, M.; Reiner, E. “Breaking down the barriers,” *Risk* **4** (September), 28-35; 1991.

Samuelson, P. “Rational theory of warrant pricing,” *Industrial Management Review*, **6** (Spring), 13 – 32; 1965.

Samuelson, P. A.; Merton, R. C. “A complete model of warrant pricing that maximizes utility,” *Industrial Management Review* **10** (Spring), 17-46; 1969.

Schnabel, J. A.; Wei, J. Z.. “Valuing takeover-contingent foreign exchange call options,” *Advances in Futures and Options Research* **7**, 223 – 236; 1994.

Schrödinger, E. “Zur Theorie der Fall- und Steigversuche an Teilchen mit Brownscher Bewegung,” *Physikalische Zeitschrift* **16**, 289-295; 1915.

Smith, C. W. “Option pricing: a review,” *Journal of Financial Economics,* **3** (January-March), 3-52; 1976.

Smithson, C. “Wonderful life” in *From Black-Scholes to black holes,* ed. R. Jacobs. Pages 23-31. London: Risk Finex; 1992.

von Smoluchowski, M. “Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen,” *Annalen der Physik* **21**, 756-780; 1906.

von Smoluchowski, M. “Experimentell nachweisbare der üblichen Thermodynamik wiedersprechende Molekularphänomene,” *Physikalische Zeitschrift* **13**, 1069-1080; 1912.

von Smoluchowski, M. “Molekulartheoretische Studien über Umkehr thermodynamisch irreversibler Vorgänge und über Wiederkehr abnormaler Zustände,” *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse*, **124**, 339-368; 1915.

von Smoluchowski, M. “Drei Vortrage über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen,” *Physikalische Zeitschrift* **17**, 557-571 and 585-599; 1916.

Sprenkle, C. M. “Warrant prices as indicators of expectations and preferences,” *Yale Economic Essays* **1**, 178 – 231; 1961.

Sprenkle, C. M. “Warrant prices as indicators of expectations and preferences,” in *The random character of stock market prices*, ed. P. H. Cootner. Pages 412 – 474. Cambridge, MA: MIT Press; 1964.

Stratonovich, R. L. “A new form of notation for stochastic integrals and equations” (in Russian), *Bulletin of Moscow University, Mathematical and Mechanical Series* **1**, 3-12; 1964.

Stratonovich, R. L. “A new representation for stochastic integrals and equations,” *SIAM Journal of Control*, **4**, 362-371; 1966.

Thorp, E. O.; Kassouf, S. T. *Beat the market*. New York, NY: Random House; 1967.

Treloar, A. E. “Outline of Biometric Analysis.” Minneapolis, MN: Burgess Publishing; 1935.

Trippi, R. R.; Chance, D. M. “Quick valuation of the Bermuda capped option,” *Journal of Portfolio Management* **20** (Fall), 93 – 99; 1993.

Wiener, N. “The average of an analytic functional,” *Proceedings of the National Academy of Sciences*, **7**, 253-260;1921a.

Wiener, N. “The average of an analytic functional and the Brownian movement,” *Proceedings of the National Academy of Sciences*, **7**, 294-298; 1921b.

Wiener, N. “Differential-space,” *Journal of Mathematics and Physics Massachusetts Institute of Technology*, **2**, 131-174; 1923.

Wilmott P.; Howison, S.; Dewynne, J. *The mathematics of financial derivatives*. Cambridge, MA: Cambridge University Press; 1997.

Zhang, P. G. “A unified formula for outside barrier options,” *Journal of Financial Engineering*, **4** , 335 – 349; 1995.

TABLE I. STOCHASTIC ORDINARY DIFFERENTIAL EQUATION

dS = μS dt + S dw

	Ito Interpretation	Stratonovich Interpretation
Fokker-Plank Equation	$\frac{\partial p}{\partial t} = -\mu \frac{\partial (Sp)}{\partial S} + \frac{\sigma^2}{2} \frac{\partial^2 (S^2 p)}{\partial S^2}$	$\frac{\partial p}{\partial t} = -(\mu + \frac{\sigma^2}{2}) \frac{\partial (Sp)}{\partial S} + \frac{\sigma^2}{2} \frac{\partial^2 (S^2 p)}{\partial S^2}$
Transition probability density function	$p(S,t) = \frac{\exp\{- \frac{[\ln(S/S_0) - (\mu - \sigma^2/2)t]^2}{2\sigma^2 t}\}}{S\sqrt{2\pi\sigma^2 t}}$	$p(S,t) = \frac{\exp\{- \frac{[\ln(S/S_0) - \mu t]^2}{2\sigma^2 t}\}}{S\sqrt{2\pi\sigma^2 t}}$
Moments	$E[S^n] = S_0^n \exp[n\mu t + n(n-1)(\sigma^2/2)t]$	$E[S^n] = S_0^n \exp[n\mu t + n^2(\sigma^2/2)t]$
Average	$E[S] = S_0 \exp(\mu t)$	$E[S] = S_0 \exp[(\mu + \sigma^2/2)t]$
Variance	$Var[S] = S_0^2 \exp(2\mu t)[\exp(\sigma^2 t) - 1]$	$Var[S] = S_0^2 \exp[2(\mu + \sigma^2/2)t][\exp(\sigma^2 t) - 1]$
Median	$Median[S] = S_0 \exp[(\mu - \sigma^2/2)t]$	$Median[S] = S_0 \exp(\mu t)$

TABLE II. STOCHASTIC ORDINARY DIFFERENTIAL EQUATION

$dy = \mu\, dt + dw$

	Ito or Stratonovich Interpretation
Fokker-Plank Equation	$\frac{\partial p}{\partial t} = - \mu \frac{\partial p}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial y^2}$
Transition probability density function	$p(y,t) = \frac{\exp\{- \frac{[y - \mu t]^2}{2\sigma^2 t}\}}{\sqrt{2\pi \sigma^2 t}}$
Moments	$E[y^n] = \sum_{m=0}^{[n/2]} \binom{n}{2m} (2\sigma^2 t)^m (\mu t)^{n-2m} \frac{\Gamma(m + 1/2)}{\sqrt{\pi}}$
Average	$E[y] = \mu t$
Variance	$Var[y] = \sigma^2 t$
Median	$Median[y] = \mu t$

TABLE III. PARAMETERS IDENTIFICATION FOR THE STOCHASTIC ORDINARY DIFFERENTAIL EQUATION

dS = μS dt + S dw

Ito Interpretation	Stratonovich Interpretation
$L(\mu,\sigma)=\prod_{i=1}^{N}\frac{\exp\{-\frac{[\ln(S_i/S_{i-1}-(\mu-\sigma^2/2)\Delta t_i]^2}{2\sigma^2\Delta t_i}\}}{S_i\sqrt{2\pi\sigma^2\Delta t_i}}$	$L(\mu,\sigma)=\prod_{i=1}^{N}\frac{\exp\{-\frac{[\ln(S_i/S_{i-1})-\mu\Delta t_i]^2}{2\sigma^2\Delta t_i}\}}{S_i\sqrt{2\pi\sigma^2\Delta t_i}}$
$\sigma^2{}_{MLE}=\frac{1}{N}\sum_{i=1}^{N}\Delta t_i\,[\frac{1}{\Delta t_i}\ln(\frac{S_i}{S_{i-1}})-\frac{1}{t_N}\ln(\frac{S_N}{S_0})]^2$	$\sigma^2{}_{MLE}=\frac{1}{N}\sum_{i=1}^{N}\Delta t_i\,[\frac{1}{\Delta t_i}\ln(\frac{S_i}{S_{i-1}})-\frac{1}{t_N}\ln(\frac{S_N}{S_0})]^2$
$\mu=\frac{1}{t_N}\ln(\frac{S_N}{S_0})+\frac{\sigma^2}{2}$	$\mu=\frac{1}{t_N}\ln(\frac{S_N}{S_0})$

TABLE IV. PARAMETERS IDENTIFACTION FOR THE STOCHASTIC ORDINARY DIFFERENTIAL EQUATION

dy = μ dt + dw

Ito or Stratonovich Interpretation
$L(\mu,\sigma) = \prod_{i=1}^{N} \frac{\exp\{-\frac{[y_i - y_{i-1} - \mu \Delta t_i]^2}{2\sigma^2 \Delta t_i}\}}{\sqrt{2\pi \sigma^2 \Delta t_i}}$
$\sigma^2{}_{MLE} = \frac{1}{N} \sum_{i=1}^{N} \Delta t_i \, [\frac{1}{\Delta t_i}(y_i - y_{i-1}) - \frac{1}{t_N} \sum_{i=1}^{N} y_i]^2$
$\mu = \frac{1}{t_N} \sum_{i=1}^{N} y_i$