

THE RISK-NEUTRAL DISTRIBUTION MODELED AS A MIXTURE OF NORMAL DISTRIBUTIONS

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INTRODUCTION

One approach used to address the significant departures of the return rates from normality is to assume that the conditional distribution of the return rates is normal whereas the unconditional distribution is not normal. The generalized autoregressive conditional heteroscedasticity models of Engle (1982) and Bollerslev (1986) belong to this category. An alternative approach is to model the unconditional distribution of the return rates using a non-normal probability density function. Generalized hyperbolic distributions were introduced by Barndorff-Nielsen (1977). In finance, Eberlein and Keller (1995) were the first to apply stochastic processes based on these distributions. The hyperbolic distribution can be presented as a normal variance-mean mixture where the mixing distribution is a generalized inverse Gaussian (Bibby and Sørensen 1997). Mixtures of normal probability density functions constitute a simplified case of hyperbolic density (Alexander and Narayanan 2001). Brigo and Mercurio (2002a; 2002b) proved the existence of a unique risk-neutral measure for the case when the price follows such a stochastic process.

MOMENTS

For a given probability density function, $f: (-\infty, +\infty) \rightarrow \mathbb{R}$, the raw moments are defined as follows:

$$\mu'_n = \int_{-\infty}^{+\infty} x^n f(x) dx \quad n = 0, 1, 2, 3, \dots$$

From the normality condition imposed on the probability density function it results that:

$$\mu'_0 = 1$$

It should be noted that the first raw moment is the average:

$$\mu'_1 = \bar{x}$$



For the same probability density function, the central moments are defined as follows:

$$\mu_n = \int_{-\infty}^{+\infty} (x - \bar{x})^n f(x) dx \quad n = 0, 1, 2, 3, \dots$$

From the normality condition imposed on the probability density function and the definition of the average it results that:

$$\begin{aligned} \mu_0 &= 1 \\ \mu_1 &= 0 \end{aligned}$$

It should be noted that the second central moment is the variance. The first raw moments can be computed as follows:

$$\begin{aligned} \mu_2' &= \mu_2 + \bar{x}^2 \\ \mu_3' &= \mu_3 + 3\mu_2 \bar{x} + \bar{x}^3 \\ \mu_4' &= \mu_4 + 4\mu_3 \bar{x} + 6\mu_2 \bar{x}^2 + \bar{x}^4 \end{aligned}$$

NORMAL DISTRIBUTION

For a normal distribution with average *av* and standard deviation *std* we have:

$$\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= std^2 \\ \mu_3 &= 0 \\ \mu_4 &= 3 std^4 \end{aligned}$$

The raw moments are as follows:

$$\begin{aligned} \mu_1' &= av \\ \mu_2' &= std^2 + av^2 \\ \mu_3' &= 3 std^2 av + av^3 \\ \mu_4' &= 3 std^4 + 6 std^2 av^2 + av^4 \end{aligned}$$



MIXTURE OF THREE NORMAL DISTRIBUTIONS

Let us consider a probability density function that is obtained as the linear combination of three normal distributions f_i :

$$f = \sum_{i=1}^3 w_i \cdot f_i \quad \text{where} \quad 0 \leq w_i \leq 1$$

From the normality condition imposed on the probability density function it results that:

$$\sum_{i=1}^3 w_i = 1$$

Let us consider three normal distributions with the following averages and standard deviations:

$$av_i = \left(r_i - \frac{\sigma_i^2}{2}\right) \cdot t \quad \text{and} \quad std_i = \sigma_i \sqrt{t} \quad i = 1, 2, 3$$

The raw moments of these three normal distributions are as follows:

$$\begin{aligned} \mu'_{1,i} &= \left(r_i - \frac{\sigma_i^2}{2}\right) \cdot t \\ \mu'_{2,i} &= \sigma_i^2 \cdot t + \left(r_i - \frac{\sigma_i^2}{2}\right)^2 \cdot t^2 \\ \mu'_{3,i} &= 3 \sigma_i^2 \cdot t^2 \cdot \left(r_i - \frac{\sigma_i^2}{2}\right) + \left(r_i - \frac{\sigma_i^2}{2}\right)^3 \cdot t^3 \\ \mu'_{4,i} &= 3 \sigma_i^4 \cdot t^2 + 6 \sigma_i^2 \cdot \left(r_i - \frac{\sigma_i^2}{2}\right)^2 \cdot t^3 + \left(r_i - \frac{\sigma_i^2}{2}\right)^4 \cdot t^4 \end{aligned}$$

From the definition of the raw moments, for a mixture of three normal distributions we obtain:

$$\begin{aligned} \mu'_1 &= \sum_{i=1}^3 w_i \cdot \left(r_i - \frac{\sigma_i^2}{2}\right) \cdot t \\ \mu'_2 &= \sum_{i=1}^3 w_i \cdot \left[\sigma_i^2 \cdot t + \left(r_i - \frac{\sigma_i^2}{2}\right)^2 \cdot t^2\right] \\ \mu'_3 &= \sum_{i=1}^3 w_i \cdot \left[3 \sigma_i^2 \cdot t^2 \cdot \left(r_i - \frac{\sigma_i^2}{2}\right) + \left(r_i - \frac{\sigma_i^2}{2}\right)^3 \cdot t^3\right] \end{aligned}$$



$$\mu_4' = \sum_{i=1}^3 w_i \cdot [3 \sigma_i^4 \cdot t^2 + 6 \sigma_i^2 \cdot (r_i - \frac{\sigma_i^2}{2})^2 \cdot t^3 + (r_i - \frac{\sigma_i^2}{2})^4 \cdot t^4]$$

From the definition of skewness and kurtosis, for a mixture of three normal distributions we obtain:

$$\begin{aligned} \text{skewness} &= \frac{\mu_3' - 3 \cdot \mu_2' \cdot \mu_1' + 2 \cdot \mu_1'^3}{(\mu_2' - \mu_1')^{\frac{3}{2}}} \\ \text{kurtosis} &= \frac{\mu_4' - 4 \cdot \mu_3' \cdot \mu_1' + 6 \cdot \mu_2' \cdot \mu_1'^2 - 3 \cdot \mu_1'^4}{(\mu_2' - \mu_1')^2} - 3 \end{aligned}$$

The Volatility of the Mixture of Three Normal Distributions

Let us consider a normal distribution with the following average and standard deviation:

$$\text{avg} = (r - y - \frac{\sigma^2}{2}) \cdot t \quad \text{and} \quad \text{stdev} = \sigma \sqrt{t}$$

Let us assume that this normal distribution matches the variance of the non-normal distribution obtained as the mixture of three normal distributions. The volatility is defined by the following equation:

$$\sigma^2 \cdot t + (r - y - \frac{\sigma^2}{2})^2 \cdot t^2 = \mu_2'$$

This equation in σ can be rewritten as follows:

$$\sigma^4 + 4 \cdot [\frac{1}{t} - (r - y)] \cdot \sigma^2 + 4 \cdot [(r - y)^2 - \frac{\mu_2'}{t^2}] = 0$$

If

$$1 - (r - y) \cdot t + \mu_2' \geq 0$$

and

$$(r - y) \cdot t - 1 + \sqrt{1 - (r - y) \cdot t + \mu_2'} \geq 0$$



then the resulting volatility is:

$$\sigma = \sqrt{2 \cdot \frac{(r - y) \cdot t - 1 + \sqrt{1 - (r - y) \cdot t + \mu_2'}}{t}}$$

THE ARBITRAGE-FREE CONDITION FOR THE MIXTURE OF THREE NORMAL DISTRIBUTIONS

If S_0 is the asset value at time $t = 0$, S_T is the asset value at time $t = T$, r is the risk-free interest rate and y is the dividend yield, then the martingale condition for an arbitrage-free economy is (Abadir and Rockinger 2003):

$$S_0 = e^{-(r-y) \cdot t} E_t[S_T]$$

where the expected value is computed based on the risk-neutral density. If the risk-neutral density is modeled as a mixture of three normal distributions, then this condition becomes:

$$S_0 = e^{-(r-y) \cdot t} \sum_{i=1}^3 w_i \cdot S_0 \cdot e^{r_i \cdot t}$$

or:

$$\sum_{i=1}^3 w_i \cdot e^{r_i \cdot t} = e^{(r-y) \cdot t}$$

If $y = 0$ it becomes identical with the no-arbitrage condition reported by Brigo et al. (2002).

PARAMETERS IDENTIFICATION FOR THE MIXTURE OF THREE NORMAL DISTRIBUTIONS

The parameters defining the linear combination of three normal distributions are w_i , r_i , and σ_i , $i = 1, 2, 3$. Based on a given set of numerical values for these parameters, we can compute σ_{Mixture} , $\text{skewness}_{\text{Mixture}}$, $\text{kurtosis}_{\text{Mixture}}$, as presented above. The mixture of normal distributions is required to:

1. Fulfill the no-arbitrage condition; and
2. Fit the observed volatility, skewness and kurtosis.

The objective function under consideration is:



$$\begin{aligned}
F = & \text{Weight}_1 \cdot \left[\frac{\ln\left(\sum_{i=1}^3 w_i \cdot e^{r_i \cdot t}\right)}{t} - (r - y) \right]^2 + \\
& \text{Weight}_2 \cdot (\sigma_{\text{Mixture}} - \sigma)^2 + \\
& \text{Weight}_3 \cdot (\text{skewness}_{\text{Mixture}} - \text{skewness})^2 + \\
& \text{Weight}_4 \cdot (\text{kurtosis}_{\text{Mixture}} - \text{kurtosis})^2
\end{aligned}$$

This function should be minimized with respect to w_i , r_i , and σ_i ($i = 1, 2, 3$) with the following conditions:

$$\sum_{i=1}^3 w_i = 1$$

$$\begin{aligned}
w_i &\geq 0, & i &= 1, 2, 3 \\
\sigma_i &> 0, & i &= 1, 2, 3
\end{aligned}$$

The minimization problem is expected to have multiple solutions. Among these solutions, we should select the solution that best fits the observed volatility smile (Alexander and Narayanan 2001).

REFERENCES

- Abadir, K. M.; Rockinger, M. Density functionals, with an option-pricing application. *Econometric Theory* **19**: 778-811; 2003.
- Alexander, C.; Narayanan, S. Option Pricing with Normal Mixture Returns. Discussion Paper 2001-10. Reading: ISMA Centre, University of Reading; 2001. accessed at <http://www.ismacentre.rdg.ac.uk/pdf/discussion/DP2001-10.pdf> on May 17, 2006
- Barndorff-Nielsen, O. E. Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society London* **A353**: 401-419; 1977.
- Bibby, B. M.; Sørensen, M. A hyperbolic diffusion model for stock prices. *Finance and Stochastics* **1**: 25-41; 1997.
- Brigo, D.; Mercurio, F. "Displaced and Mixture Diffusions for Analytically-Tractable Smile Models". In Geman, H.; Madan, D. B.; Pliska, S. R.; Vorst, A. C. F. (Eds.) *Mathematical Finance - Bachelier Congress 2000*, pages 151-174. Heidelberg: Springer Finance; 2002a.
- Brigo, D.; Mercurio, F. Lognormal-Mixture Dynamics and Calibration to Market Volatility Smiles. *International Journal of Theoretical & Applied Finance* **5** (4): 427-446; 2002b.



Brigo, D.; Mercurio, F.; Sartorelli, G. Lognormal-mixture dynamics under different means. 2002. accessed at <http://www.fabiomercurio.it/qfinternal.pdf> on May 16, 2006.

Bollerslev, T. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* **31** (3): 307-327; 1986.

Eberlein, E.; Keller, U. Hyperbolic distributions in finance. *Bernoulli* **1**: 281—299; 1995.

Engle, R. F. Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica* **50** (4): 987-1008; 1982.

