

DERIVATIVE VALUATION – GRAM-CHARLIER

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The Hermite polynomials form the basis of a Hilbert space and may be used to get an expansion of the probability density function. Usually, this series is called Gram-Charlier. For practical purposes, only the first few terms of this expansion are taken into consideration. The resulting truncated series may be viewed as the normal probability density function multiplied by a polynomial that accounts for the effects of departure from normality. The Gram-Charlier series uses the moments of the real distribution. The Edgeworth series is similar to Gram-Charlier but uses cumulants instead of moments. Although the series are equivalent, for computational purposes the Gram-Charlier series seems to perform better than the Edgeworth series (Johnson et al., 1994).

This approach was introduced in financial economics by Jarrow and Rudd (1982), and it has been applied by Madan and Milne (1994), Longstaff (1995), Abken et al. (1996a ; 1996b), Brenner and Eom (1997), Knight and Satchell (1997), Backus et al. (1997), Corrado and Su (1997).

Non-normal skewness and kurtosis in option-implied distributions are found to contribute significantly to the phenomenon of volatility smile. For the S&P 500 Index, historical rates of return have a mean in the range 0.009 to 0.13, a standard deviation in the range 0.11 to 0.35, a skewness in the range of (-1.12) to 0.15, and a kurtosis in the range 0.22 to 8.92. On the average, the implied standard deviation is 0.1162, the implied skewness is (-1.68), and the implied kurtosis is 2.39 (Corrado and Su, 1997).

We use the following notations: Φ denotes the Laplace function (i.e., the normal cumulative probability function), ϕ denotes the normal probability density function, S denotes the stock price, K denotes the strike price, T denotes the expiration date, t denotes the current date, r denotes the risk free interest rate, q denotes the yield, σ denotes the volatility, γ_1 denotes the skewness coefficient and γ_2 denotes the kurtosis coefficient.



Taking into account the skewness and kurtosis without neglecting the higher powers of the volatility, the price of a European call is:

$$\begin{aligned}
 \text{Call} = & S e^{-q(T-t)} - \sigma^3 (T-t)^{3/2} \frac{\gamma_1}{6} - \sigma^4 (T-t)^2 \frac{\gamma_2}{24} \Phi(d) - \\
 & - K e^{-r(T-t)} \Phi(d - \sigma \sqrt{T-t}) + \\
 & + S e^{-q(T-t)} \sigma \sqrt{T-t} \phi(d) \left\{ (2 \sigma \sqrt{T-t} - d) \frac{\gamma_1}{6} + \right. \\
 & + \left. [3 \sigma^2 (T-t) - 3 \sigma \sqrt{T-t} d + d^2 - 1] \frac{\gamma_2}{24} \right\} + \\
 & + S e^{-q(T-t)} - \sigma^3 (T-t)^{3/2} \frac{\gamma_1}{6} - \sigma^4 (T-t)^2 \frac{\gamma_2}{24} \bullet \\
 & \bullet \left[\sigma^3 (T-t)^{3/2} \frac{\gamma_1}{6} + \sigma^4 (T-t)^2 \frac{\gamma_2}{24} \right] \Phi(d)
 \end{aligned}$$

where:

$$d = \frac{\left[\ln\left(\frac{S}{K}\right) + (r-q)(T-t) + \frac{\sigma^2(T-t)}{2} - \sigma^3(T-t)^{3/2} \frac{\gamma_1}{6} - \sigma^4(T-t)^2 \frac{\gamma_2}{24} \right]}{\sigma \sqrt{T-t}}$$

If we discard the third and fourth powers of the volatility, this formula is identical to the one used by Backus et al. (1997).

Although the Gram-Charlier (and Edgeworth) expansion allows for additional flexibility over the normal probability density function because it introduces the skewness and kurtosis of the empirical distribution as parameters, this expansion has the drawback of yielding negative values for certain skewness-kurtosis parameters because it is a polynomial approximation. Rubinstein (1998) provides approximate skewness-kurtosis values for which the Gram-Charlier and Edgeworth expansions do not provide negative values. Jondeau and Rockinger (1999; 2001) use the method developed by Barton and Dennis (1952) to characterize the boundary delimiting the domain in the skewness-kurtosis space over which the Gram-Charlier expansion is positive. To ensure the positivity constraints, the kurtosis must be between zero and four. For each acceptable kurtosis value there is a symmetrical interval for the skewness. This interval can be computed numerically. However, all acceptable skewness values are between (-1.05) and (+1.05). Therefore, the positivity constraints require that the Gram-Charlier approach be used for relatively moderate departures from normality.

The price of call/put European options can be used to identify the skewness and kurtosis. The identification procedure can or cannot take into account the positivity constraints of Jondeau and Rockinger (1999; 2001). Jondeau and Rockinger (2001) suggest that when the unconstrained identification yields skewness-kurtosis values for which the positivity constraints are violated we



are faced with a model misspecification. Theoretically this difficulty can be overcome by adding further terms in the expansion. There are several reasons why this approach may fail:

1. The additional parameters render more difficult the search for the domain where the approximation is positive.
2. The additional parameters may shrink the domain where the approximation is positive. This is the case of the Edgeworth model (Dennis 1952; Rubinstein 1998).
3. The additional parameters complicate the identification procedure that may become numerically unstable.
4. Increasing the number of terms in the approximation may induce multicollinear parameters (Corrado and Su 1996; Jarrow and Rudd 1982). “[A]dding the terms b_5 or b_6 to skewness and kurtosis procedures leads to highly unstable parameter estimates” (Corrado and Su 1996).

Therefore, for practical purposes it is recommended to limit the approximation to the first four moments.

Testing the identification procedure, Jondeau and Rockinger (2001) mention that while the estimates of the first two moments are very stable, the estimates for skewness and kurtosis tend to differ by a relatively large percentage from the true values. The nearer the empirical distribution is to the normal distribution, the less likely it is that an adequate parametric approximation can be found. This finding is similar to the one of Bowman and Shenton (1973): “[T]here is the paradox that, the nearer to normality the theoretical distribution is, the less likely it is that a normal mixture fit can be found.”

In conclusion, the Gram-Charlier estimates are of better quality when the distribution is significantly different with respect to the normal one. It is therefore recommended to check first for departures from normality. Only if there is a significant departure from normality should we resort to the Gram-Charlier approximation. The Gram-Charlier approximation should be used with caution whenever the Jondeau and Rockinger (1999; 2001) positivity constraints are violated.

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