BLACK-SCHOLES ILLUSTRATED

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The input data are the stock price (**S**), the strike price (**K**), the time until expiration (**t**), the volatility (**σ**), the risk-free interest-rate (**r**), and the yield (**y**). The natural logarithm of the stock price on the expiration date (**ln(S)**) follows a normal distribution with the following average and standard deviation:

$$average = \ln(S) + \left(r - y - \frac{\sigma^2}{2}\right) \bullet t$$
$$standard\ deviation = \sigma\sqrt{t}$$

Of course, the stock price on the expiration date **(S)** follows the corresponding log-normal distribution. Its probability density function is denoted as **f(S)**, while its cumulative probability function is denoted as **F(S)**.

The value of a call is computed as follows:

$$call = e^{(-rt)} \int_K^{\infty} (S - K) \bullet f(S)\, dS$$

The above formula can be rewritten as:

$$call = \int_0^{\infty} e^{(-rt)} \bullet IntrinsicValue(S) \bullet f(S)\, dS$$

where:

$$IntrinsicValue(S) = \mathrm{MAX}[0, (S - K)]$$

Taking as the upper limit of integration a large but finite number, the call option valuation can be approximated as follows:

$$call = e^{(-rt)} \bullet IntrinsicValue\left(\frac{S_1 + S_2}{2}\right)[F(S_2) - F(S_1)] +$$
$$+ \sum_{i=2}^{N-1} e^{(-rt)} \bullet IntrinsicValue\left(\frac{S_i + S_{i+1}}{2}\right)[F(S_{i+1} - F(S_i)]$$

where $\mathbf{S_i}$**, i = 1, 2, ..., N** is a partition of the integration interval $\mathbf{(0, S_{MAX})}$:

$$0 = S_1 < S_2 < S_3 < \ldots S_{i-1} < S_i < S_{i+1} < \ldots < S_{N-1} < S_N = S_{\mathrm{MAX}}$$

Therefore, the steps are as follows:

1. Get the upper and lower limits for each interval $\mathbf{(S_i, S_{i+1})}$
2. For each interval, compute the mid-point
3. Compute the intrinsic value for each mid-point
4. Discount the intrinsic values
5. Compute the probability attached to each interval
6. For each interval, multiply the discounted intrinsic valued by the probability attached to that interval
7. Sum all results from 6

NOTE 1
The output array has 7 columns:

- Lower limit of the interval
- Upper limit of the interval
- Mid-point of the interval
- Intrinsic value
- Discounted intrinsic value
- Interval probability
- Discounted intrinsic value multiplied by the interval probability

Each row corresponds to a given interval. The first two columns identify the interval specifying its lower and upper limits. The third column provides the mid-point, a value that is considered as representative for that interval. The intrinsic value is computed based on the mid-point value. The next column computes the discounted intrinsic value. The interval probability lists the probability for the stock to be between the lower and upper limits of the interval. The last column is simply the product of the previous two columns.

NOTE 2

The first interval is always (0, K). The stock cannot have negative values . Therefore, the probability listed for this interval is Probability [0 < S < K] or simply Probability [S < K].

$$
\begin{aligned}
\textbf{Delta} &= \exp(-y \bullet t) \bullet N(d1) = \\
&= \exp(-y \bullet t) \bullet N\{ [\text{average} -(\ln(K) - \text{variance})]/\text{Standard deviation}\} = \\
&= \exp(-y \bullet t) \bullet \text{Probability } [\ln(S) > \ln(K) - \text{variance}] = \\
&= \exp(-y \bullet t) \bullet \text{Probability } [\ln(S) > \ln(K) - \sigma \bullet \sigma \bullet t] = \\
&= \exp(-y \bullet t) \bullet \text{Probability } [S > K \bullet \exp(-\sigma \bullet \sigma \bullet t)] = \\
&= \exp(-y \bullet t) \bullet \{1 - \text{Probability } [S < K \bullet \exp(-\sigma \bullet \sigma \bullet t)]\} \geq \\
&\geq \exp(-y \bullet t) \bullet \{1 - \text{Probability } [S < K]\}
\end{aligned}
$$

or conversely

Probability $[S < K]\} \geq 1 -$ Delta $\bullet \exp(y \bullet t)$

Therefore, the first probability listed in the output array is not expected to equal 1 - Delta.

NOTE 3

The first precursor of the Black-Scholes model is the Bachelier (1900) model assuming a Brownian motion for the stock prices which results in a normal distribution of the stock prices. This model predicts negative stock values. Sprenkle improved the model assuming a Brownian motion for the return rate (instead of the stock price) and assuming an average return rate that can be non-zero.

NOTE 4

The basic assumptions of the Black-Scholes model that were relaxed:

Assumption	**Relaxed by**
No dividends	Merton (1973)
No taxes or transaction costs	Ingersoll(1976)
Constant interest rates	Merton (1973)
No penalties for short sales	Thorpe (1973)
Market operates continuously	Merton (1976)
Share price is continuous	Cox &Ross (1976)
Return distribution is normal	Jarrow & Rudd (1982)

The numerical methods can be classified into three categories:

- Tree models (i.e. binomial)
- Finite difference models
- Monte Carlo models