BETA ANALYSIS

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CAPITAL ASSET PRICING MODEL

The Sharpe-Lintner-Mossin mean-variance equilibrium model of exchange (Sharpe 1964; Lintner 1965 a, b; Treynor 1965; Mossin 1966), commonly called the Capital Asset Pricing Model (CAPM), is based on the following assumptions:

1. All investors are averse to risk;
2. All investors maximize the expected utility of terminal wealth for a single period;
3. All investors have identical decision horizons and homogeneous expectations regarding investment opportunities;
4. All investors are able to choose among portfolio solely on the basis of expected rates of return and expected volatilities of the rates of return;
5. All transaction costs and taxes are zero;
6. All assets are infinitely divisible;
7. The capital market is in equilibrium.

According to the CAPM, the ***expected*** rates of return for one period are related as follows:

$$E[r] - r_f = \beta \cdot (E[r_{INDEX}] - r_f)$$

where:

r	=	rate of return of the financial instrument
$\mathbf{r_f}$	=	risk-free interest rate
$\mathbf{r_{INDEX}}$	=	rate of return of the index.

and **E[ ]** denotes the expected value. The ***realized*** rates of return follow a similar regression. For a financial instrument (e.g., stock) its rate of return is regressed on the rate of return of an index using the following formula (Sharpe 1991):

$$r \quad = \quad \alpha \quad + \quad \beta \cdot r_{INDEX}$$

The graphical representation of this relationship is termed a security or portfolio's *characteristic line*. The CAPM implies that (Sharpe 1991):

$$\alpha = (1 - \beta) \cdot r_f$$

Much confusion has arisen regarding the relationship between the equilibrium results of the CAPM and the underlying relationships among security returns. The CAPM makes no assumptions about the return generating process. Therefore, its results are consistent with any such process (Sharpe 1991).

While the initial version of the CAPM was quite parsimonious (Sharpe 1991), it has been extended and adapted to incorporate real-world phenomena such as: returns in real terms (Lintner 1969); taxation (Brennan 1970); lack of riskless assets (Black 1972); investors' concern with future investment opportunities (Merton 1973); more general classes of utility functions (Rubinstein 1974); the skewness of the return distribution (Kraus and Litzenberger 1976); transaction costs (Levy 1978); investors' preferences for consumption (Breeden 1979); market segmentation (Merton 1987); short sales restrictions (Markovitz 1990).

PARAMETERS ESTIMATION

The data are collected on a discrete basis at the moments $\mathbf{t_0 < t_1 < t_2 < t_3 < ... < t_N}$. The corresponding values collected for the financial instrument are $\mathbf{S_0, S_1, S_2, S_3, ..., S_N}$, while the corresponding values collected for the index are $\mathbf{Index_0, Index_1, Index_2, Index_3, ..., Index_N}$.

The rates of return are computed as follows:

$$r_{i+1} = \ln(\frac{S_{i+1}}{S_i}) / (t_{i+1} - t_i) \qquad i = 0,1,\ldots,N-1$$

$$r_{INDEX,i+1} = \ln(\frac{INDEX_{i+1}}{INDEX_i}) / (t_{i+1} - t_i) \qquad i = 0,1,\ldots,N-1$$

The regression parameters are identified through the minimization of the following objective function:

$$F = \sum_{i=1}^{N} w_i \cdot (r_i - \alpha - \beta \cdot r_{INDEX,i})^2$$

where $\mathbf{w_i} \geq \mathbf{0}$. The solution of the minimization problem is:

$$\hat{\beta} = \frac{S_{r_{INDEX} r}}{S_{r_{INDEX} r_{INDEX}}} \quad and \quad \hat{\alpha} = \bar{r} - \hat{\beta} \cdot \bar{r}_{INDEX}$$

where:

$$\bar{r}_{INDEX} = \left(\sum_{i=1}^{N} w_i \cdot r_{INDEX,i} \right) / \left(\sum_{i=1}^{N} w_i \right)$$

$$\bar{r} = \left(\sum_{i=1}^{N} w_i \cdot r_i \right) / \left(\sum_{i=1}^{N} w_i \right)$$

$$S_{r_{INDEX} r_{INDEX}} = \sum_{i=1}^{N} w_i \cdot (r_{INDEX,i} - \bar{r}_{INDEX})^2$$

$$S_{r_{INDEX} r} = \sum_{i=1}^{N} w_i \cdot (r_{INDEX,i} - \bar{r}_{INDEX}) \cdot (r_i - \bar{r})$$

The mean square residual is:

$$MS_e = \left(S_{rr} - \frac{S^2{}_{r_{INDEX} r}}{S_{r_{INDEX} r_{INDEX}}} \right) / (N - 2)$$

where:

$$S_{rr} = \sum_{i=1}^{N} w_i \cdot (r_i - \bar{r})^2$$

For each parameter separately, the marginal confidence interval can be computed based on the best estimate and its standard deviation using the Student t-statistic (Walpole and Myers1978):

$$\text{Confidence interval for the slope:}$$
$$\hat{\beta} \pm t_{\frac{\alpha}{2}, N-2} \cdot \sqrt{MS_e / S_{r_{INDEX} r_{INDEX}}}$$

Confidence interval for the intercept:

$$\hat{\alpha} \pm t_{\frac{\alpha}{2},N-2} \cdot \sqrt{MS_e \cdot \left(\frac{1}{\sum_{i=1}^{N} w_i} + \frac{\bar{r}^2{}_{INDEX}}{S_{r_{INDEX} r_{INDEX}}} \right)}$$

This approach does not take into account the correlation between the estimates of the two parameters. The joint confidence region for both parameters is an ellipse that is computed using the Fisher F-statistic (Bates and Watts 1988):

$$(\alpha - \hat{\alpha})^2 \cdot \sum_{i=1}^{N} w_i + 2 \cdot (\alpha - \hat{\alpha}) \cdot (\beta - \hat{\beta}) \cdot \sum_{i=1}^{N} w_i \cdot r_{INDEX,i} +$$

$$+ (\beta - \hat{\beta})^2 \cdot \sum_{i=1}^{N} w_i \cdot r^2{}_{INDEX,i} = 2 \cdot MS_e \cdot F_{\alpha,2,N-2}$$

For a given value of $\mathbf{r_{INDEX}}$ there is a prediction interval for a new “one at a time” realization of **r** (Walpole and Myers1978)**:**

$$\hat{\alpha} + \hat{\beta} \cdot r_{INDEX} \pm t_{\frac{\alpha}{2},N-2} \cdot \sqrt{MS_e \cdot \left(1 + \frac{1}{\sum_{i=1}^{N} w_i} + \frac{(\bar{r}_{INDEX} - r_{INDEX})^2}{S_{r_{INDEX} r_{INDEX}}} \right)}$$

Additionally, for a given value of $\mathbf{r_{INDEX}}$ there is a narrower confidence band for the regressed value (i.e., average of realizations) of **r** (Walpole and Myers1978)**:**

$$\hat{\alpha} + \hat{\beta} \cdot r_{INDEX} \pm t_{\frac{\alpha}{2},N-2} \cdot \sqrt{MS_e \cdot \left(\frac{1}{\sum_{i=1}^{N} w_i} + \frac{(\bar{r}_{INDEX} - r_{INDEX})^2}{S_{r_{INDEX} r_{INDEX}}} \right)}$$

The average rates of return are computed as follows:

$$R \quad = \quad \ln(\frac{S_N}{S_0}) \,/\, (t_N - t_0)$$

$$R_{INDEX} = \ln(\frac{INDEX_N}{INDEX_0}) / (t_N - t_0)$$

The usual BETA analysis is performed using all rates of return. The UP BETA analysis is similar to the usual BETA analysis, but it uses only the values (r_i, $r_{INDEX,i}$) that fulfill the following conditions (Ang and Chen 2002):

$$r_i > R \quad and \quad r_{INDEX,i} > R_{INDEX}$$

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Similarly, the DOWN BETA analysis uses only the values (r_i, $r_{INDEX,i}$) that fulfill the following conditions (Ang and Chen 2002):

$$r_i < R \quad and \quad r_{INDEX,i} < R_{INDEX}$$

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It should be noted that some values (r_i, $r_{INDEX,i}$) that are used for BETA analysis may be discarded by both the UP BETA and DOWN BETA analyses.

RISK MEASURES

A practical problem regarding portfolio management is how to evaluate the performance of risky investments. Any “portfolio performance” measure has to address the following two aspects:

1. Maximization of the rate of return; and
2. Minimization of the risk.

The Treynor (1965) measure is defined as follows:

$$\text{Treynor Ratio} = \frac{r - r_f}{\beta}$$

This measure is similar to the Sharpe (1966) ratio:

$$\text{Sharpe Ratio} = \frac{r - r_f}{\sigma}$$

where **σ** is the volatility.

The Jensen (Jensen 1968) measure is defined as follows:

$$\text{Jensen Measure} = \alpha + (\beta - 1) \cdot r_f$$

The UP and DOWN versions of these measures can be obtained using the **α** and **β** estimates from the BETA analysis.

MANAGING VALUE AT RISK (VaR)

Incremental Value at Risk (IVaR) is a standard tool to identify strategies that enhance return and control risk. Garman (1996; 1997), Dowd (1999; 2000) and Mina (2002) were among the first researchers and practitioners to identify the relevance of IVaR for discriminating acceptable investments. A straightforward computation of IVaR requires the computation of VaR before and after the potential portfolio change. Due to the non-linearity of VaR, these computations may be time-expensive and unacceptable for real-time decision making. User-friendly approximations for IVaR can be obtained using the BETA analysis.

Risk adding can be performed when purchasing a new asset using additional money (i.e., when the amount of invested money is increased). When risk is measured using VaR, a risk reduction is indicated by a negative IVaR (Tasche and Tibiletti 2003):

$$\mathrm{I}VaR \approx \beta \cdot VaR \cdot a$$

where:

β	=	Beta coefficient of the new asset with respect to the present portfolio
a	=	position considered to be bought **(a>0)** or sold **(a<0)**

If the Beta coefficient (of the new asset with respect to the present portfolio) is negative, then buying that asset will act as a risk diversifier. Vice versa, for a position already contained in the portfolio, a negative Beta coefficient (of this asset with respect to the portfolio) signals that selling it will act as a risk contributor. Therefore, as intuition suggests, adding a super-defensive position (i.e., it goes in the opposite direction to that of the portfolio) or selling a conservative/aggressive position (i.e., it goes in the same direction as that of the portfolio) will reduce risk.

Risk pooling can be performed when purchasing a new asset without using additional money (i.e., when the weights of the extant assets are reduced). When risk is measured using VaR, the risk change is (Tasche and Tibiletti 2003):

$$\mathrm{I}VaR \approx (\beta - 1) \cdot VaR \cdot a$$

In the case of risk pooling, the condition for an acceptable strategy is not the sign of **β** (as in the case of risk adding), but the sign of **(β – 1)**. Even if **β** is positive (but less than one), pooling of at least a small portion of the new asset under consideration may be advisable for risk reduction; such a defensive position goes in the same direction like the portfolio, but at a smaller speed. Except the case of a super-aggressive position (i.e., **β > 1**), pooling will reduce risk.

The fact that the "risk pooling" conditions are looser than those for "risk adding" should not be a surprise. By virtue of the favorable diversification effect, the acceptance of pooling in the portfolio a sufficiently long string of single-rejected risks is "rational." However, the eventual acceptance of adding single-rejected risks is questionable (Tasche and Tibiletti 2003).

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